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Ph.D. Thesis

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Kolmogorov's model of turbulence - mathematical analysis

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Abstract

In the thesis we investigate the local and global existence of regular solutions to Kolmogorov's two-equation model of turbulence. The local existence results are obtained for the initial data with different assumptions on their regularity. First, the periodic domain is considered with initial data from H^2. Secondly, the existence of solutions is shown for torus and data from H^s, $s > \frac{d}{2}$. Obtained solutions are unique. The proof of the existence requires the commutator estimate for the Bessel potential J^s, which is adapted from a well-known result for $\mathbb{R}^d$. The global existence of a regular solution is shown under a smallness condition imposed on the initial data. The condition is formulated in such a way to ensure the absorption of high-order terms by the diffusive terms.

Keywords: Kolmogorov's model of turbulence, existence of solution, uniqueness of solution, local in time existence, fractional Sobolev spaces, commutator estimates

Streszczenie

W pracy zbadano istnienie lokalnych i globalnych w czasie, regularnych rozwiązań dwurównaniowego modelu turbulencji Kołmogorowa. Istnienie lokalnych w czasie regularnych rozwiązań jest pokazane przy różnych założeniach na regularność danych początkowych. Najpierw pokazane jest istnienie rozwiązań na periodycznej dziedzinie i danych początkowych z przestrzeni H^2. Następnie udowodnione zostaje istnienie rozwiązań na torusie z danymi początkowymi pochodzącymi z przestrzeni H^s, gdzie $s > \frac{d}{2}$. Przeprowadzany dowód wymaga oszacowania komutatora dla potencjału Bessela. Wynik ten, dobrze znany dla przypadku $\mathbb{R}^d$, jest pokazany dla przypadku torusa. Istnienie globalnych w czasie, regularnych rozwiązań jest pokazane przy dodatkowym założeniu na małość danych początkowych. Warunek dobrany jest tak by gwarantować absorpcję wyrazów wyższych rzędów przez człon dyfuzyjny.

Słowa kluczowe: model turbulencji Kołmogorowa, istnienie rozwiązań, jednoznaczność rozwiązań, lokalne w czasie rozwiązania, ułamkowe przestrzenie Sobolewa, oszacowania komutatorów

Contents

Introduction

The flow of the fluid can be described using various partial differential equations such as the Navier-Stokes, Euler and Stokes systems. In the engineering practice, the most commonly used model is one given by the Navier-Stokes equations. In the basic form, the model describes the flow of a viscous, incompressible, isothermal fluid with constant density. The Navier-Stokes equations have been extensively studied both from the theoretical and numerical sides. The theoretical research concentrates on establishing or disproving the regularity/uniqueness of solutions to the Navier-Stokes system. So far the problem remains open. On the other hand, the numerical research shows that fluid flows characterised by high Reynold's number (turbulent flow) are difficult to simulate. The difficulty arises from the fact that a turbulent flow is characterised by chaotic fluctuations of the velocity and pressure fields. Thus to simulate the fluid flow properly the computational mesh should be very fine and the simulation's time-step very small (see [52]). For example, the simulation of the planar turbulent channel with Reynold's number 10 000 would require 50 million CPU-hours (see [40] and [24]). Those difficulties with the Navier-Stokes equations motivate the exploration of alternative formulations of hydrodynamics.

In 1941 A. N. Kolmogorov in [27] proposed the following system of equations describing the flow of the turbulent fluid:

$$v_{,t} + \operatorname{div}(v \otimes v) - \nu_0 \operatorname{div}\left(\frac{b}{\omega} D(v)\right) = -\nabla p, \tag{1}$$

$$\omega_{,t} + \operatorname{div}(\omega v) - \kappa_1 \operatorname{div}\left(\frac{b}{\omega} \nabla \omega\right) = -\kappa_2 \omega^2, \tag{2}$$

$$b_{,t} + \operatorname{div}(b v) - \kappa_3 \operatorname{div}\left(\frac{b}{\omega} \nabla b\right) = -b\omega + \kappa_4 \frac{b}{\omega} |D(v)|^2, \tag{3}$$

$$\operatorname{div} v = 0, \tag{4}$$

where D - denotes the symmetric gradient, v - the mean velocity field, b - $2/3$ of the mean turbulent kinetic energy, ω - the dissipation rate of the mean turbulent kinetic energy (also referred to as the scale of turbulence), p - the sum of the pressure and b. The constants ν_0, κ_1, κ_2, κ_3, κ_4 are given positive material parameters. The equations are equipped with the initial conditions

$$v_{|t=0} = v_0, \quad \omega_{|t=0} = \omega_0, \quad b_{|t=0} = b_0 \tag{5}$$

and boundary conditions, which will be established later. In considering the Navier-Stokes system difficulties arise due to high oscillations. The main idea behind the formulation is to consider a smoothen-out (averaged) velocity field. However, to account for an information lost (about the instantaneous velocity V and the instantaneous pressure P) due to the averaging process, additional quantities have to be introduced. Instead of tracking the fluctuation's velocity $v' = V - v$ directly, the mean kinetic energy of v' (i.e. the mean turbulent kinetic energy) is considered. Additionally, the proposed model introduces the dissipation rate, which accounts for the transfer of the turbulent kinetic energy into the internal thermal energy. We see that the increase of the kinetic energy causes an increase of the artificial (turbulent) viscosity $\frac{b}{\omega}$. It is known from the theory of the Navier-Stokes equation that a larger viscosity lengthens the existence time of regular solutions. Such an artificial viscosity also improves the numerical properties of the equation.

Nowadays, the ideas introduced by Kolmogorov are used in the development of new turbulence models such as $k-\varepsilon$, and $k-\omega$ (see [12], [48], [10], [51]). Each of these models is based on some artificial viscosity dependent on other mean flow quantities. These turbulence models are inherently prone to inaccuracies due to the introduced averaging. Thus based on the application, the choice of the turbulence model can significantly affect the prediction. To understand the averaging process better, we will provide the derivation of a part of Kolmogorov's system from the Euler equations. Let us note that the used procedure can also be applied for viscous flows i.e. such described by the Navier-Stokes equations.

The simplest idea that would decrease the apparent fluctuations of solutions is to consider the average value of the velocity and of the pressure. This is the case in Kolmogorov's approach. To this end let us introduce averaging operator $f \mapsto \overline{f}$. Now, let us decompose the flow's velocity V and the pressure P in the following way:

$$V(x,t) = v(x,t) + v'(x,t), \quad P(x,t) = \widetilde{p}(x,t) + p'(x,t),$$

where v, $\widetilde{p}$ are the time-averaged values and v', p' account for fluctuations around mean values. We additionally require the following conditions to hold:

$$\overline{v} = v, \qquad \overline{v'} = 0, \qquad \overline{\widetilde{p}} = \widetilde{p}, \qquad \overline{p'} = 0.$$

We substitute the decomposed functions into the Euler system and we get (for details see chapter 2 of [51]):

$$\partial_t(v + v') + \operatorname{div}\left((v + v') \otimes (v + v')\right) + \nabla(\widetilde{p} + p') = 0.$$

By applying the average operator to the equation we obtain

$$\partial_t v + \operatorname{div}(v \otimes v) + \nabla\widetilde{p} = -\operatorname{div}\left(\overline{v' \otimes v'}\right).$$

The last term on the right-hand side can be approximated by the Boussinesq approximation (see [51])

$$-\overline{v' \otimes v'} = \nu_T(\nabla v + \nabla^T v) - bI,$$

where ν_T denotes the turbulent viscosity. In the considered case of Kolmogorov's system we set $\nu_T = \frac{\nu_0}{2}\frac{b}{\omega}$. Finally, we obtain

$$\partial_t v + \operatorname{div}(v \otimes v) - \nu_0 \operatorname{div}\left(\frac{b}{\omega} Dv\right) + \nabla p = 0, \tag{6}$$

where $p = \widetilde{p} + b$. This way we derived equation (1) based on the Euler equation. However, we see that to close the system we need to introduce additional equations for ω and b. The derivation of equations (2), (3) is more complicated and requires additional postulates besides the Boussinesq approximation. For further details see [51] and [48].

Recently, the research concerning the mathematical analysis of Kolmogorov's model has accelerated. In [8] authors showed the existence of a weak solution to Kolmogorov's turbulence model. It relies on the introduction of a new variable $E = |v|^2/2+b$ representing the total energy in the system. It allows for the replacement of b-equation with an equation depending on E:

$$\partial_t E + \operatorname{div}(v(E+p)) - 2\nu_0 \operatorname{div}\left(\frac{\kappa_3 b}{\kappa_4 \omega}\nabla b + \frac{b}{\omega}D(v)v\right) + \frac{2\nu_0}{\kappa_4}b\omega = 0.$$

The main benefit of $E-$equation lies in the absence of $\frac{b}{\omega}|D(v)|^2$ term, which enables passage to the limit with the approximate solution. Additionally, it is worth of noting that the developed methodology allows for b_0 such that: $b_0 > 0$ and $\ln b_0 \in L^1(\Omega)$. In the article [36] the authors consider the system (1)-(4) in a periodic domain. The authors first consider the approximate problem with an additional p-Laplacian term. This allows for deriving additional estimates for the symmetric gradient of solutions. After passing to the limit the authors obtain a global-in-time weak solution. However, due to the presence of the strongly nonlinear term $\frac{b}{\omega}|D(v)|^2$, the weak form of equation (3) has to be corrected by a positive measure μ. The assumption on the initial value of b is that it has to be uniformly positive. In [16] the authors consider the 1D system motivated by Kolmogorov's system structure (with the omitted pressure term). The system of equations is considered in the periodic setting. First, the authors prove the local-in-time existence of solutions for the initial data such that $(v_0, \omega_0, b_0) \in H^2$, $b_0 \geqslant 0$, $\sqrt{b_0} \in H^2$ and ω_0 is strictly positive. This choice of initial data means that the diffusion coefficient may vanish. Also, they prove the existence of a class of smooth initial data, for which a finite-time blow-up occurs. More precisely, the blow-up occurs in a finite time provided: $b_0(0) = 0$, v_0 is odd with respect to 0, ω_0 and b_0 are even with respect to 0, $\partial_x v_0(0) < 0$ and $\sqrt{b_0} \in H^3$. In [17] the authors continue the work from [16] however for the modified (yet still relevant to the analysis of Kolmogorov's model) system of equations. The authors prove additional conditions which cause the blow-up in a finite time. These conditions include the second derivative of turbulent kinetic energy. There are also some developments regarding the theoretical analysis of other turbulence models, however less fruitful (in terms of an obtained regularity) due to mathematically less advantageous structure of equations: [33], [13], [15], [14], [38], [35].

The aim of the thesis is to show the existence of the regular solutions to Kolmogorov's two-equation model of turbulence. First, we show the existence of a regular solution in a small time interval. The obtained solution's norms may potentially blow up after a certain finite time. The minimal existence time of the solution is determined by the initial data and the model's parameters. The local-in-time existence is studied in two settings: for initial data from the space H^2 and from H^s. In the second case, we have to assume that initial data are regular enough i.e. $s > \frac{d}{2}$. We also prove that the obtained solutions are unique. The second part of the thesis concentrates on showing the existence of (nontrivial) global-in-time solutions under the smallness condition imposed on the initial data. The additional condition's purpose is to ensure the absorption of higher-order terms by the diffusive term. It is worth noting that the regularity analysis of Kolmogorov's system was not considered in the literature.

The thesis is comprised of 5 chapters. Now we will give a brief description of the contents of each chapter. Chapter 1 provides an information about function spaces, that will be used throughout the thesis. Additionally, certain useful, yet simple estimates are provided. In Chapter 2 the local-in-time-existence of a solution is studied for the initial data from the space H^2. The detailed result is formulated in Theorem 2.1.1. The result is then used in the next chapter. In Chapter 3 the existence of global in time, regular solutions is shown under a smallness condition imposed on initial data. The basic idea behind the considerations is to show that with the help of the smallness condition, the solutions (provided by results from Chapter 2) can be extended indefinitely in time. In Chapter 4, a local-in-time-existence of the solution is proven for initial data from $H^s(\mathbb{T}^d)$, where $s > \frac{d}{2}$. Also, it is shown that such solutions are unique. These results improve the result given in Chapter 2. Also to obtain the existence result, Appendix contents are utilised. In Chapter 5 the existence of global weak solutions on the torus is shown. The main purpose of this chapter is to provide a better understanding of the approach presented in [8]. The Appendix concentrates on adapting a proof of a commutator estimate presented in [26] for the case of the torus. Finally, in the Summary, the conclusions are formulated as well as possible directions for a continuation of work on the subject.

Chapter 1

Function spaces and auxiliary lemmas

In this chapter, we introduce the notation used throughout the thesis. Additionally, the basic inequalities are also given.

Let Ω be a domain (to be specified in each chapter), $T > 0$ and $\Omega^T = \Omega \times (0,T)$. Let $(X, \|\cdot\|)$ be a Banach space and $1 \leqslant p \leqslant \infty$. Throughout the thesis, we will denote the Bochner space by $L^p(0,T;X)$. The space is equipped with the following norm

$$\|u\|_{L^p(0,T;X)} := \left(\int_0^T \|u(t)\|_X^p \, dt\right)^{\frac{1}{p}} \qquad \text{for } 1 \leqslant p < \infty,$$

$$\|u\|_{L^\infty(0,T;X)} := \text{ess sup}_{t\in[0,T]} \|u(t)\|_X .$$

Moreover, the space of continuous functions from $[0,T]$ to X will be denoted by $C([0,T];X)$. The space is equipped with the following norm

$$\|u\|_{C([0,T];X)} := \max_{t\in[0,T]} \|u(t)\|_X .$$

Let us consider $f : \mathbb{R}^d \to \mathbb{R}$. By $\nabla^k f$ we denote the k-dimensional matrix comprised of elements $\frac{\partial^k f}{\partial^{k_1} x_1 ... \partial^{k_d} x_d}$, where $k_i \geqslant 0$ for $i = 1, ..., d$ and $\sum_{i=1}^d k_i = k$. Based on this we define

$$\left\|\nabla^k f\right\|_{L^2(\Omega)}^2 = \sum_{(k_1,...,k_d)\in S_k} \left\|\frac{\partial^k f}{\partial^{k_1} x_1 ... \partial^{k_d} x_d}\right\|_{L^2(\Omega)}^2 ,$$

where $S_k = \{(k_1, ..., k_d) \in \mathbb{N}^d : \sum_{i=1}^d k_i = k\}$.

1.1. Function spaces on $\Pi_{i=1}^3(0,L_i)$

Let $\Omega = \Pi_{i=1}^3(0,L_i)$, $r \geqslant 1$ and $k \in \mathbb{N}$. By $W^{k,r}(\Omega)$ we denote the space of restrictions to Ω of the functions, which belong to the space

$$\{u \in W^{k,r}_{loc}(\mathbb{R}^3): \ u(\cdot + kL_i e_i) = u(\cdot) \quad \text{for} \quad k \in \mathbb{Z}, \ i = 1,2,3\},$$

where $\{e_i\}_{i=1}^3$ forms the standard basis in $\mathbb{R}^3$. We shall denote by $\|\cdot\|_{k,2}$ the norm in the Sobolev space, i.e.

$$\|f\|_{k,2} = (\|\nabla^k f\|_2^2 + \|f\|_2^2)^{\frac{1}{2}}, \tag{1.1}$$

where $\|\cdot\|_2$ is L^2 norm on Ω. Additionally, we define $W^{1,r}_{\mathrm{div}}(\Omega)$ in the following way:

$$W^{1,r}_{\mathrm{div}}(\Omega) = \{v \in \left[W^{1,r}(\Omega)\right]^3: \ \mathrm{div}\, v = 0 \text{ in } \Omega, \ \int_\Omega v dx = 0\}.$$

Dual spaces of $W^{1,r}$ and $W^{1,r}_{\mathrm{div}}$ will be denoted, respectively, in the following way:

$$W^{-1,r'}(\Omega) := \left(W^{1,r}(\Omega)\right)^*, \qquad W^{-1,r'}_{\mathrm{div}}(\Omega) := \left(W^{1,r}_{\mathrm{div}}(\Omega)\right)^*,$$

where $\frac{1}{r} + \frac{1}{r'} = 1$. Let $1 \leqslant p < \infty$. By $\|\cdot\|_p$ and $\|\cdot\|_{1,p}$, we denote classical norms in $L^p(\Omega)$ and $W^{1,p}(\Omega)$, respectively:

$$\|f\|_p = \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}}, \qquad \|f\|_{1,p} = \left(\|f\|_p^p + \sum_{i=1}^3 \|\partial_{x_i} f\|_p^p\right)^{\frac{1}{p}}.$$

Now, we define the following transformation:

$$\langle \cdot, \cdot \rangle : W^{-1,r} \times W^{1,r'} \to \mathbb{R}$$

such that for $f \in W^{-1,r}(\Omega)$ and $g \in W^{1,r'}(\Omega)$, where $\frac{1}{r} + \frac{1}{r'} = 1$, we have

$$\langle f, g \rangle := f(g).$$

Thus, we can define the norm in dual spaces of Sobolev spaces:

$$\|f\|_{-1,r} = \sup_{\varphi \in W^{1,r'}(\Omega): \|\varphi\|_{W^{1,r'}(\Omega)} = 1} |\langle f, \varphi \rangle| .$$

Also, for $f \in L^r(\Omega)$ and $g \in L^{r'}(\Omega)$, where $\frac{1}{r} + \frac{1}{r'} = 1$, we define $(\cdot, \cdot)$ in the following way:

$$(f, g) = \int_\Omega f(x)g(x)dx.$$

Additionally, we define the space

$$L^2_{\mathrm{div}}(\Omega) := \overline{W^{1,2}_{\mathrm{div}}(\Omega)}^{\|\cdot\|_2},$$

where the right-hand side denotes the closure of the space $W^{1,2}_{\mathrm{div}}(\Omega)$ in $L^2(\Omega)$ norm. Moreover, let

$$L^r_0(\Omega) := \{v \in L^r(\Omega) : \int_\Omega v dx = 0\}.$$

Finally, we define the space that will be useful for considerations related to the kinetic turbulent energy b:

$$\begin{aligned} \varepsilon = \{b \in L^\infty\left(0, T, L^1(\Omega)\right) &: b > 0 \text{ almost everywhere in } \Omega^T, \\ \ln b \in L^\infty(0, T, L^1(\Omega)), b &\in L^\lambda(0, T, W^{1,\lambda}(\Omega)) \ \ \forall \lambda \in [1, 2)\}. \end{aligned} \tag{1.2}$$

If $m \in \mathbb{N}$, then by $\mathcal{V}^m$ we denote the space of restrictions to Ω of the functions, which belong to the space

$$\{u \in H^m_{loc}(\mathbb{R}^3) : \ u(\cdot + kL_i e_i) = u(\cdot) \quad \text{for} \quad k \in \mathbb{Z}, \ \ i = 1, 2, 3\}, \tag{1.3}$$

where $\{e_i\}_{i=1}^3$ form a standard basis in $\mathbb{R}^3$. Also we define

$$\dot{\mathcal{V}}^m_{\mathrm{div}} = \{v \in \mathcal{V}^m : \ \operatorname{div} v = 0, \ \int_\Omega v dx = 0\}. \tag{1.4}$$

For the convenience we also introduce the following space

$$\mathcal{X}(T) = L^2(0, T; \dot{\mathcal{V}}^3_{\mathrm{div}}) \times L^2(0, T; \mathcal{V}^3)) \times (L^2(0, T; \mathcal{V}^3) \cap (H^1(0, T; H^1(\Omega)))^5. \tag{1.5}$$

1.1.1. The Gagliardo-Nireberg inequalities

In this subsection, we collect the special cases of the Gagliardo-Nirenberg inequalities used in the paper (for the original formulation and proof see [20], [39], [18]). Here, the constant c depends only on Ω and we assume that f is a periodic function on Ω, it is sufficiently regular to make the right-hand side finite. Firstly, we recall

$$\|\nabla f\|_4^2 \leqslant c \|\nabla f\|_2 \left\|\nabla^3 f\right\|_2 . \tag{1.6}$$

The lower order term (say, L^2 norm) can be omitted, because $\int_\Omega \nabla f dx = 0$, $\int_\Omega \nabla^2 f dx = 0$ and from the Poincaré inequality for functions with the vanishing mean we get

$$\|\nabla f\|_2^2 = \|\nabla f\|_2 \|\nabla f\|_2 \leqslant C_1 \|\nabla f\|_2 \left\|\nabla^2 f\right\|_2 \leqslant C_2 \|\nabla f\|_2 \left\|\nabla^3 f\right\|_2 ,$$

where C_1, C_2 depends only on Poincaré constant for Ω. Next, we have

$$\|f\|_3^2 \leqslant c \|\nabla f\|_2 \|f\|_2 , \quad \text{if} \quad \int_\Omega f dx = 0, \tag{1.7}$$

$$\|f\|_6 \leqslant c \|\nabla f\|_2 , \quad \text{if} \quad \int_\Omega f dx = 0, \tag{1.8}$$

$$\|\nabla f\|_6^2 \leqslant c \left\|\nabla^3 f\right\|_2 \|\nabla f\|_2 , \tag{1.9}$$

$$\|\nabla f\|_4^2 \leqslant c \left\|\nabla^3 f\right\|_2 \|\nabla f\|_{\frac{3}{2}} , \tag{1.10}$$

$$\|f\|_\infty \leqslant c(\left\|\nabla^2 f\right\|_2 + \|f\|_1), \tag{1.11}$$

$$\|f\|_\infty \leqslant c \left\|\nabla^2 f\right\|_2 , \quad \text{if} \quad \int_\Omega f dx = 0, \tag{1.12}$$

$$\|f\|_{\frac{3}{2}} \leqslant c \|\nabla f\|_{\frac{3}{2}}^{\frac{1}{2}} \|f\|_1^{\frac{1}{2}} + c\|f\|_1, \tag{1.13}$$

where c depends only on Ω.

1.2. Function spaces on $\mathbb{R}^d$

Now we will recall function spaces defined on $\mathbb{R}^d$. Provided definitions and facts are mainly used to derive analogous statements in the case of $\mathbb{T}^d$.

Definition 1.2.1 (see Section 2.2.2 in [47]). Let $s \in \mathbb{R}$ and $1 < p < \infty$. Then we define the Bessel-potential space in the following way:

$$H_p^s\left(\mathbb{R}^d\right) = \{f \in S' : \|f\|_{H_p^s(\mathbb{R}^d)} = \left\|F^{-1}\left[(1+|x|^2)Ff\right]\right\|_p < \infty\},$$

where F denotes the Fourier transform and S' is the space of tempered distributions.

Now we will list useful facts related to the introduced space.

Lemma 1.2.1 (see Theorem 13.8.1 in [32]). *Let $s > \frac{d}{2}$ and $f \in H^s(\mathbb{R}^d)$. Then, function f is continuous and there exists constant $C = C(s,d)$ independent of f such that*

$$\|f\|_{L^\infty(\mathbb{R}^d)} \leqslant C\,\|f\|_{H^s(\mathbb{R}^d)}\,.$$

Theorem 1.2.2 (see Lemma X4 in [26]). *Let $s \geqslant 0$. Then there exists a constant $C = C(s,d)$ such that $\forall f,g \in H^s(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ the following inequality holds:*

$$\|fg\|_{H^s(\mathbb{R}^d)} \leqslant C\left(\|f\|_{L^\infty(\mathbb{R}^d)}\|g\|_{H^s(\mathbb{R}^d)} + \|f\|_{H^s(\mathbb{R}^d)}\|g\|_{L^\infty(\mathbb{R}^d)}\right).$$

Lemma 1.2.3 (see Theorem 5.5 in [3] or Section 3.1 in [46]). *Let $s > \frac{d}{2}$. Assume that F is a smooth function on $\mathbb{R}$ with $F(0) = 0$. Then there exists $C = C(s,d)$ independent of u and F such that:*

$$\|F(u)\|_{H^s(\mathbb{R}^d)} \leqslant C\,\|F'\|_{C^{\lceil s\rceil}}\left(1 + \|u\|_{L^\infty(\mathbb{R}^d)}\right)^{\lceil s\rceil}\|u\|_{H^s(\mathbb{R}^d)}\,.$$

1.3. Function spaces on d-dimensional torus

Let us start with recalling the definitions of spaces set on $\mathbb{T}^d = [0,1)^d$.

Definition 1.3.1 (see Remark 3.1.5 in [42] or Section 3.2 and 3.5 in [2]). Let $\{u_n\}_{n=1}^\infty \subset C^\infty(\mathbb{T}^d)$, $u \in C^\infty(\mathbb{T}^d)$. We say that $u_n \to u$ in $C^\infty(\mathbb{T}^d)$ if $\partial^\alpha u_n \to \partial^\alpha u$ uniformly for all $\alpha \in \mathbb{N}_0$. By $\mathcal{D}'(\mathbb{T}^d)$ we denote the space of continuous linear functionals on $C^\infty(\mathbb{T}^d)$.

Definition 1.3.2 (see Definition 3.1.6 in [42]). Let $S(\mathbb{Z}^d)$ denote the space of rapidly decaying functions from $\mathbb{Z}^d$ to $\mathbb{C}$. That is, $\varphi \in S(\mathbb{Z}^d)$ if for any $k < \infty$ there exists a constant $C_{\varphi,k}$ such that

$$|\varphi(\xi)| \leqslant \frac{C_{\varphi,k}}{(1+|\xi|^2)^{k/2}}.$$

The topology on $S(\mathbb{Z}^d)$ is given by the seminorms p_k, where $k \in \mathbb{N}_0$ and

$$p_k(\varphi) = \sup_{\xi \in \mathbb{Z}^d} \left(1+|\xi|^2\right)^{\frac{k}{2}} |\varphi(\xi)|.$$

Then, a sequence $\{\varphi_n\}_{n=1}^{\infty} \subset S(\mathbb{Z}^d)$ converges to the function $\varphi \in S(\mathbb{Z}^d)$ iff

$$p_k(\varphi_n - \varphi) \overset{n\to\infty}{\longrightarrow} 0 \quad \text{for all } k \in \mathbb{N}_0.$$

By $S'(\mathbb{Z}^d)$ we denote the space of continuous linear functionals on $S(\mathbb{Z}^d)$.

Definition 1.3.3 (see Definition 3.1.8 in [42])**.** Toroidal Fourier transform $\mathcal{F}_{\mathbb{T}^d} = (f \mapsto \hat{f}) : C^\infty(\mathbb{T}^d) \to S(\mathbb{Z}^d)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{T}^d} f(x) e^{-2\pi i x\cdot\xi} dx.$$

Inverse toroidal Fourier transform $\mathcal{F}_{\mathbb{T}^d}^{-1} = (h \mapsto \check{h}) : S(\mathbb{Z}^d) \to C^\infty(\mathbb{T}^d)$ is given by

$$\check{h}(x) = \sum_{\xi \in \mathbb{Z}^d} h(\xi) e^{i2\pi x\cdot\xi}.$$

Definition 1.3.4 (see Definition 3.1.27 in [42])**.** Fourier transform extends to the mapping $\mathcal{F}_{\mathbb{T}^d} : \mathcal{D}'(\mathbb{T}^d) \to S'(\mathbb{Z}^d)$ by the formula

$$\langle \hat{u}, \varphi \rangle := \langle u, \iota \circ \check{\varphi} \rangle,$$

where $u \in \mathcal{D}'(\mathbb{T}^d)$, $\varphi \in S(\mathbb{Z}^d)$ and ι is defined by $(\iota \circ \psi)(x) = \psi(-x)$.

Definition 1.3.5. Let $s \in \mathbb{C}$. The Bessel potential J^s on the torus is defined as follows

$$(J^s f)(x) = \sum_{k \in \mathbb{Z}^d} \left(1 + 4\pi^2 |k|^2\right)^{s/2} e^{2\pi i x\cdot k} \hat{f}(k),$$

where $\hat{f}$ denotes Fourier transform of f.

Now, let us recall the definition of fractional inhomogeneous Sobolev spaces on torus $H^s_p(\mathbb{T}^d)$.

Definition 1.3.6 (see chapter 3.5.4 in [43]). Let s be a real number, $p \in (1,\infty)$. The inhomogeneous Sobolev Space $H^s_p(\mathbb{T}^d)$ is defined as follows

$$H^s_p(\mathbb{T}^d) = \left\{u \in D'(\mathbb{T}^d) : \|u\|_{H^s_p} = \|J^s u\|_p < \infty\right\}.$$

Moreover, based on the orthogonality of $\{e^{2\pi kx}\}_{k\in\mathbb{Z}^d}$ in $L^2(\mathbb{T}^d)$ the following characterisation holds

$$H^s(\mathbb{T}^d) = \left\{u \in D'(\mathbb{T}^d) : \|u\|^2_{H^s_2} = \sum_{k\in\mathbb{Z}^d} \left(1 + 4\pi^2|k|^2\right)^s |\hat{u}(k)|^2 < \infty\right\}.$$

Moreover, we introduce the following notation

$$H^s_{\mathrm{div}}(\mathbb{T}^d) = \left\{u \in [H^s(\mathbb{T}^d)]^d : \operatorname{div} u = 0\right\}.$$

To simplify further expressions we introduce also notation:

$$(f,g)_{H^s} = \left(J^s f, \overline{J^s g}\right)_{L^2(\mathbb{T}^d)}.$$

Now, we will recall some known facts concerning fractional Sobolev spaces on the torus.

Lemma 1.3.1 (see [9], [22], [26], [23]). *Let $s \geqslant 0$ and $p \in (1,\infty)$, $p_1, p_2, q_1, q_2 \in (1,\infty]$ satisfy*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}.$$

Let $f, g \in C^\infty(\mathbb{T}^d)$. Then there exists $C = C(s,d,p,p_1,p_2,q_1,q_2)$ independent of f and g such that the following inequality holds:

$$\|J^s(fg)\|_p \leqslant C\left(\|J^s f\|_{p_1}\|g\|_{q_1} + \|f\|_{p_2}\|J^s g\|_{q_2}\right).$$

Lemma 1.3.2 (see chapter 2.8.3 in [47])**.** *Let $p \in (1, \infty)$, $s > d/p$ and $f, g \in H_p^s(\mathbb{T}^d)$. Then, $fg \in H_p^s(\mathbb{T}^d)$ and there exists a constant $C > 0$, independent of f and g such that*

$$\|fg\|_{H_p^s(\mathbb{T}^d)} \leqslant C \|f\|_{H_p^s(\mathbb{T}^d)} \|g\|_{H_p^s(\mathbb{T}^d)} .$$

Lemma 1.3.3 (see the proof in Appendix)**.** *Suppose that $s > 0$, $p, p_2, p_4 \in (1, \infty)$ and $p_1, p_3 \in (1, \infty]$ such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Let $f, g \in C^\infty(\mathbb{T}^d)$, then there exists a constant $C = C(s, d)$ independent of f and g such that

$$\|[J^s, f]\, g\|_{L^p(\mathbb{T}^d)} \leqslant C \left(\|\nabla f\|_{L^{p_1}(\mathbb{T}^d)} \left\|J^{s-1} g\right\|_{L^{p_2}(\mathbb{T}^d)} + \|g\|_{L^{p_3}(\mathbb{T}^d)} \|J^s f\|_{L^{p_4}(\mathbb{T}^d)} \right),$$

where $[J^s, f]\, g := J^s(fg) - f J^s g$.

Lemma 1.3.4 (See Lemma 2.5(ii) in [11])**.** *Let $s > \frac{d}{2}$ and $f \in H^s\left(\mathbb{T}^d\right)$. Then, the function f is continuous and there exists a constant $C = C(s, d)$ independent of f such that*

$$\|f\|_\infty \leqslant C \|f\|_{H^s} .$$

Lemma 1.3.5 (see Theorem 5.5 in [3] or Section 3.1 in [46])**.** *Let $s > \frac{d}{2}$. Assume that G is a smooth function on $\mathbb{R}$ with $G(0) = 0$. Then there exists C independent of $f \in H^s\left(\mathbb{T}^d\right)$ and G such that:*

$$\|G(f)\|_{H^s} \leqslant C \|G'\|_{C^{\lceil s \rceil}} \left(1 + \|f\|_\infty\right)^{\lceil s \rceil} \|f\|_{H^s} .$$

Lemma 1.3.6. *Let $s > \frac{d}{2}$. Assume that G is a smooth function on $\mathbb{R}$ with $G'(0) = 0$. Then there exists C independent of $u, v \in H^s(\mathbb{T}^d)$ and G such that:*

$$\|G(u) - G(v)\|_{H^s} \leqslant C \|G''\|_{C^{\lceil s \rceil}} \|u - v\|_{H^s} \left(1 + \|u\|_{H^s} + \|v\|_{H^s}\right)^{\lceil s \rceil + 1} .$$

Proof. The lemma is a direct consequence of Lemma 1.3.5. Proceeding as in [1] Corollary 2.66, we see that

$$G(u) - G(v) = (u - v)\int_0^1 G'(u + \tau(v-u))d\tau,$$

which can be understood classically due to v, u both being continuous functions (see Lemma 1.3.4). By applying $H^s\left(\mathbb{T}^d\right)$ norm to the both sides and using Lemma 1.3.2 we get

$$\|G(u) - G(v)\|_{H^s} \leqslant \|u - v\|_{H^s} \left\|\int_0^1 G'(u + \tau(v-u))d\tau\right\|_{H^s}.$$

Next, we may change the order of the norm and integral to get

$$\|G(u) - G(v)\|_{H^s} \leqslant \|u - v\|_{H^s} \int_0^1 \|G'(u + \tau(v-u))\|_{H^s} d\tau.$$

As $G'(0) = 0$ we may apply Lemma 1.3.5 to the term under integral

$$\begin{aligned}&\|G(u) - G(v)\|_{H^s} \\ &\qquad \leqslant C\,\|G''\|_{C^{\lceil s\rceil}} \|u - v\|_{H^s} \int_0^1 \left(1 + \|u + \tau(v-u)\|_\infty\right)^{\lceil s\rceil} \|u + \tau(v-u)\|_{H^s} d\tau.\end{aligned}$$

We may estimate the right-hand side using the triangle inequality. We get

$$\begin{aligned}&\|G(u) - G(v)\|_{H^s} \\ &\qquad \leqslant C\,\|G''\|_{C^{\lceil s\rceil}} \|u - v\|_{H^s} \left(1 + \|u\|_\infty + \|v\|_\infty\right)^{\lceil s\rceil} \left(\|u\|_{H^s} + \|v\|_{H^s}\right).\end{aligned}$$

By using Lemma 1.3.4 we obtain the desired inequality. □

Lemma 1.3.7. *Let* $f : \mathbb{T}^d \to \mathbb{C}$ *be such that* $f \in H^{s+1}(\mathbb{T}^d)$*. Then:*

$$\|f\|^2_{H^{s+1}} = \|\nabla f\|^2_{H^s} + \|f\|^2_{H^s}.$$

Proof. By simple calculations we get the assertion of the lemma:

$$
\begin{aligned}
\|\nabla f\|_{H^s}^2 &= \sum_{j=1}^{d} \sum_{k\in\mathbb{Z}^d} \left(1+4\pi^2|k|^2\right)^s \left|\left(\widehat{\frac{\partial f}{\partial x_j}}\right)(k)\right|^2 \\
&= \sum_{j=1}^{d} \sum_{k\in\mathbb{Z}^d} \left(1+4\pi^2|k|^2\right)^s 4\pi^2|k_j|^2 \left|\widehat{f}(k)\right|^2 \\
&= \sum_{k\in\mathbb{Z}^d} \left(\left(1+4\pi^2|k|^2\right)^{s+1} - \left(1+4\pi^2|k|^2\right)^s\right) \left|\widehat{f}(k)\right|^2 \\
&= \|f\|_{H^{s+1}}^2 - \|f\|_{H^s}^2 .
\end{aligned}
$$

□

Lemma 1.3.8 (See Lemma 2.5(i) in [11]). *Let $p \in (1,\infty)$ and let $\mu, \nu \in \mathbb{R}$ be such that $\nu \leqslant \mu$. Then $H_p^{\mu}(\mathbb{T}^d) \hookrightarrow H_p^{\nu}(\mathbb{T}^d)$.*

Lemma 1.3.9 (See Lemma 2.5(iii) in [11]). *Let $p, q \in (1,\infty)$ and let $\mu, \nu \in \mathbb{R}$ be such that $\nu \leqslant \mu$ and*

$$\mu - \frac{d}{p} = \nu - \frac{d}{q}.$$

Then $H_p^{\mu}(\mathbb{T}^d) \hookrightarrow H_q^{\nu}(\mathbb{T}^d)$.

Remark 1.3.10. *References of Lemmas 1.3.2 and 1.3.5 are provided for $\mathbb{R}^d$ domain. By following the argument presented in Section 2.3.1 of [11], those formulations can be adapted for $\mathbb{T}^d$ case by considering the extension operator $H_p^s(\mathbb{T}^d) \ni f \to \phi\widetilde{f} \in H_p^s(\mathbb{R}^d)$, where ϕ is a smooth, compactly supported function defined on $\mathbb{R}^d$, such that $\phi_{|[0,1)^d} = 1$ and*

$$\widetilde{f}(x) = f(x - \lfloor x \rfloor),$$

where $\lfloor x \rfloor = (\lfloor x_1 \rfloor, \ldots, \lfloor x_d \rfloor)$, ($\lfloor\cdot\rfloor$ - the floor function). Then, it is clear that

$$\|f\|_{W^{k,p}(\mathbb{T}^d)} \leqslant \left\|\phi\widetilde{f}\right\|_{W^{k,p}(\mathbb{R}^d)} \leqslant C\,\|f\|_{W^{k,p}(\mathbb{T}^d)} . \tag{1.14}$$

From the complex interpolation (see e.g.: Theorem 2.6 in [34]), we can deduce the analogous inequality for the fractional spaces, i.e.

$$\|f\|_{H^s_p(\mathbb{T}^d)} \leqslant \left\|\phi\widetilde{f}\right\|_{H^s_p(\mathbb{R}^d)} \leqslant C\,\|f\|_{H^s_p(\mathbb{T}^d)}\,. \tag{1.15}$$

Considering Lemma 1.3.1 for the function $\phi\widetilde{f}$ and using (1.15) yields the needed assertion. The proof of the Lemma 1.3.5 is more complicated and we shall give more details.

Proof. Let ψ, ϕ be smooth, compactly supported functions defined on $\mathbb{R}^d$, such that $\psi_{|[0,1]^d} \equiv 1$ and $\phi_{|\sup\psi} \equiv 1$. Also let us observe that $\widetilde{G(f)} = G(\widetilde{f})$. Now, using Lemmas 1.2.2, 1.2.1, 1.2.3 and the fact that $G(0) = 0$ we can write

$$\begin{aligned}\left\|\psi\widetilde{G(f)}\right\|_{H^s(\mathbb{R}^d)} &= \left\|\psi G(\widetilde{f})\right\|_{H^s(\mathbb{R}^d)} = \left\|\psi G(\phi\widetilde{f})\right\|_{H^s(\mathbb{R}^d)} \leqslant C\,\|\psi\|_{H^s(\mathbb{R}^d)}\left\|G(\phi\widetilde{f})\right\|_{H^s(\mathbb{R}^d)}\\ &\leqslant C\left\|G(\phi\widetilde{f})\right\|_{H^s(\mathbb{R}^d)} \leqslant C\left\|G'\right\|_{C^{\lceil s\rceil}}\left(1+\left\|\phi\widetilde{f}\right\|_{L^\infty(\mathbb{R}^d)}\right)^{\lceil s\rceil}\left\|\phi\widetilde{f}\right\|_{H^s(\mathbb{R}^d)}.\end{aligned}$$

We can easily estimate the both sides using (1.15) to get

$$\|G(f)\|_{H^s(\mathbb{T}^d)} \leqslant C\left\|G'\right\|_{C^{\lceil s\rceil}}\left(1+\|f\|_{L^\infty(\mathbb{T}^d)}\right)^{\lceil s\rceil}\|f\|_{H^s(\mathbb{T}^d))}\,.$$

□

Lemma 1.3.11. *Let $f \in H^{s+1}\left(\mathbb{T}^d\right)$ and $s > \frac{d}{2}$. Then we have*

$$\|\nabla f\|_\infty \leqslant C\,\|f\|_{H^s(\mathbb{T}^d)}^{\frac{1}{2}\left(s-\frac{d}{2}\right)}\,\|f\|_{H^{s+1}(\mathbb{T}^d)}^{1-\frac{1}{2}\left(s-\frac{d}{2}\right)} \quad \text{for } s \in \left(\frac{d}{2}, \frac{d}{2}+1\right] \tag{1.16}$$

and

$$\|\nabla f\|_\infty \leqslant C\,\|f\|_{H^s} \quad \text{for } s \in \left(\frac{d}{2}+1, \infty\right).$$

Proof. First we concentrate on the case $s \in \left(\frac{d}{2}, \frac{d}{2}+1\right]$. We see that from Lemma 1.3.4 it follows

$$\|\nabla f\|_{L^\infty(\mathbb{T}^d)} \leqslant C\,\|\nabla f\|_{H^{\frac{1}{2}\left(s+\frac{d}{2}\right)}(\mathbb{T}^d)}\,.$$

We see that

$$\frac{1}{2}\left(s+\frac{d}{2}\right)=\frac{1}{2}\left(s-\frac{d}{2}\right)(s-1)+\left(1-\frac{1}{2}\left(s-\frac{d}{2}\right)\right)s$$

and thus we may use the interpolation inequality to get

$$\|\nabla f\|_{L^\infty(\mathbb{T}^d)} \leqslant C\,\|\nabla f\|_{H^{s-1}(\mathbb{T}^d)}^{\frac{1}{2}\left(s-\frac{d}{2}\right)}\,\|\nabla f\|_{H^{s}(\mathbb{T}^d)}^{1-\frac{1}{2}\left(s-\frac{d}{2}\right)}.$$

Thus by Lemma 1.3.7 we obtain

$$\|\nabla f\|_{L^\infty(\mathbb{T}^d)} \leqslant C\,\|f\|_{H^{s}(\mathbb{T}^d)}^{\frac{1}{2}\left(s-\frac{d}{2}\right)}\,\|f\|_{H^{s+1}(\mathbb{T}^d)}^{1-\frac{1}{2}\left(s-\frac{d}{2}\right)}.$$

If $s\in\left(\frac{d}{2}+1,\infty\right)$ we have

$$\|\nabla f\|_{L^\infty(\mathbb{T}^d)} \leqslant C_1\,\|\nabla f\|_{H^{s-1}(\mathbb{T}^d)} \leqslant C_2\,\|f\|_{H^{s}(\mathbb{T}^d)},$$

which follows from Lemma 1.3.4 and 1.3.7. □

1.4. Special function

In later parts of the thesis, we utilise the existence of certain kinds of functions. Let us set $b_{\min}>0$, $0<\omega_{\min}\leqslant\omega_{\max}$. Next, we define the following auxiliary functions

$$b_{\min}^t=\frac{b_{\min}}{(1+\kappa_2\omega_{\max}t)^{\frac{1}{\kappa_2}}},\qquad \omega_{\min}^t=\frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t},\qquad \omega_{\max}^t=\frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}. \tag{1.17}$$

We will justify the existence of functions Ψ_t, Φ_t such that

$$\Psi_t(x)=\begin{cases}\frac{1}{2}b_{\min}^t & \text{for}\quad x<\frac{1}{2}b_{\min}^t,\\ x & \text{for}\quad x\geqslant b_{\min}^t,\end{cases} \tag{1.18}$$

and

$$\Phi_t(x)=\begin{cases}\frac{1}{2}\omega_{\min}^t & \text{for}\quad x<\frac{1}{2}\omega_{\min}^t,\\ x & \text{for}\quad x\in[\omega_{\min}^t,\omega_{\max}^t],\\ 2\omega_{\max}^t & \text{for}\quad x>2\omega_{\max}^t,\end{cases} \tag{1.19}$$

We further require that the functions Ψ_t, Φ_t would also satisfy

$$0 \leqslant \Psi_t'(x) \leqslant c_1, \ \ |\Psi_t^{(n)}(x)| \leqslant c_n(b_{\min}^t)^{n-1} \ \text{ for } x \in \mathbb{R}, \tag{1.20}$$

$$0 \leqslant \Phi_t'(x) \leqslant c_1, \ \ |\Phi_t^{(n)}(x)| \leqslant c_n(\omega_{\min}^t)^{n-1} \ \text{ for } x \in \mathbb{R}, \tag{1.21}$$

where, $c_n > 0$ is a constant independent of x and t.

The function Ψ_t may be defined as follows. We set $f(x) = e^{-1/x}$ for $x > 0$ and zero elsewhere. Then we set

$$\eta(x) = \frac{1}{c}\int_0^x f(y)f(1-y)dy,$$

where $c = \int_0^1 f(y)f(1-y)dy$. The function η is a smooth function, which vanishes for negative x and is equal to one for $x > 1$. Next, we put

$$h(x) = (1-\eta(x))f(x) + \eta(x)x.$$

We see that $h(x) = 0$ for $x < 0$ and $h(x) = x$ for $x > 1$. Thus, it is clear that

$$\forall n \in \mathbb{N}_+ \ \exists \widetilde{c}_n > 0 \ \ \text{such that } \forall x \in \mathbb{R} \quad |h^{(n)}(x)| \leqslant \widetilde{c}_n.$$

Now we will verify that function h is non-decreasing. In fact, we only need to check if for $x \in (0,1)$

$$h'(x) = (1-\eta(x))e^{-1/x}\frac{1}{x^2} + \eta(x) + \eta'(x)(x - e^{-1/x})$$

is non-negative. Indeed, let us recall that $0 \leqslant \eta(x) \leqslant 1$ and that η is a non-decreasing function. Also $\forall x \in (0,1)$ we have $x \geqslant e^{-1/x}$. Finally, we define

$$\Psi_t(x) = \frac{b_{\min}^t}{2} + \frac{b_{\min}^t}{2} h\left(\frac{2}{b_{\min}^t}\left(x - \frac{b_{\min}^t}{2}\right)\right). \tag{1.22}$$

It is clear that for the defined function both (1.18) and (1.20) hold. Now we define Φ_t in the following way

$$\Phi_t(x) = \begin{cases} \frac{\omega_{\min}^t}{2} + \frac{\omega_{\min}^t}{4} h\left(\frac{4}{\omega_{\min}^t}\left(x - \frac{\omega_{\min}^t}{2}\right)\right) & \text{for} \quad x < \frac{\omega_{\min}^t + \omega_{\max}^t}{2} \\ 2\omega_{\max}^t - \frac{3\omega_{\max}^t}{4} h\left(\frac{4}{3\omega_{\max}^t}\left(2\omega_{\max}^t - x\right)\right) & \text{for} \quad x \geqslant \frac{\omega_{\min}^t + \omega_{\max}^t}{2} \end{cases}. \tag{1.23}$$

We see that for $\Phi_t(x) = x$ for $x \in (\frac{3}{4}\omega^t_{\min}, \frac{5}{4}\omega^t_{\max})$. Thus it is clear that Φ_t is smooth and that both (1.19) and (1.21) hold. Also in Chapter 2 we will need functions such that

$$\psi_t(x) = \begin{cases} 0 & \text{for} \quad x < \frac{1}{2}b^t_{\min}, \\ x & \text{for} \quad x \geqslant b^t_{\min}, \end{cases} \quad , \quad \phi_t(x) = \begin{cases} 0 & \text{for} \quad x < \frac{1}{2}\omega^t_{\min} \\ x & \text{for} \quad x \geqslant \omega^t_{\min} \end{cases} \tag{1.24}$$

and

$$\psi_t(x) \leqslant x \quad \text{for} \quad x \geqslant 0, \quad 0 \leqslant \psi'_t(x) \leqslant c_0 \quad \text{for} \quad x \in \mathbb{R}, \tag{1.25}$$

$$\phi_t(x) \leqslant x \quad \text{for} \quad x \geqslant 0, \quad 0 \leqslant \phi'_t(x) \leqslant c_0 \quad \text{for} \quad x \in \mathbb{R} \tag{1.26}$$

for some constant c_0. Let us define $\widetilde{h}(x) = \eta(x)(x+1)$. The functions ψ_t, ϕ_t can be defined as follows:

$$\psi_t(x) = \frac{b^t_{\min}}{2}\widetilde{h}\left(\frac{2}{b^t_{\min}}\left(x - \frac{b^t_{\min}}{2}\right)\right), \qquad \phi_t(x) = \frac{\omega^t_{\min}}{2}\widetilde{h}\left(\frac{2}{\omega^t_{\min}}\left(x - \frac{\omega^t_{\min}}{2}\right)\right). \tag{1.27}$$

Clearly, the both functions are non-decreasing. Also by recalling $\eta(x) \leqslant 1$ we see that for $x \geqslant 0$ we have

$$\psi_t(x) = \frac{b^t_{\min}}{2}\eta\left(\frac{2}{b^t_{\min}}\left(x - \frac{b^t_{\min}}{2}\right)\right)\left(\frac{2}{b^t_{\min}}\left(x - \frac{b^t_{\min}}{2}\right) + 1\right) \leqslant x.$$

Chapter 2

Local in time solution for H^2 initial data

In this chapter we prove the existence of local in-time solutions to Kolmogorov's turbulence model. The existence result is attained for H^2 initial data in the periodic setting. The detailed formulation of the result is given in Theorem 2.1.1. In the proof of the theorem, the Galerkin method is used. First, the approximation of Kolmogorov's system is constructed. Next, uniform estimates of solution are provided. This enables the passage to the limit in an approximated problem. Finally, the bounds for ω and b are proven. The result is published in [31].

2.1. Notation and main result.

Assume that $\Omega = \prod_{i=1}^{3}(0, L_i)$, L_i, $T > 0$ and $\Omega^T = \Omega \times (0,T)$. We shall consider the problem (1)-(5) in $\Omega^T = \Omega \times (0,T)$. Constants $\nu_0, \kappa_1, \ldots, \kappa_4$ are positive. For simplicity, we assume further that all constants except of κ_2 are equal to one. The reason is that the constant κ_2 plays an important role in the a priori estimates.
We shall show the local-in-time existence of a regular solution of problem (1)-(5) under some assumption imposed on the initial data. Namely, suppose that $v_0 \in \dot{\mathcal{V}}^2_{\mathrm{div}}$, ω_0, $b_0 \in \mathcal{V}^2$ for which there exist positive numbers $b_{\min}$, $\omega_{\min}$, $\omega_{\max}$ such that

$$0 < b_{\min} \leqslant b_0(x), \tag{2.1}$$

$$0 < \omega_{\min} \leqslant \omega_0(x) \leqslant \omega_{\max} \tag{2.2}$$

on Ω. Additionally, we define

$$b^t_{\min} = \frac{b_{\min}}{(1+\kappa_2\omega_{\max}t)^{\frac{1}{\kappa_2}}}, \quad \omega^t_{\min} = \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t},$$
$$\omega^t_{\max} = \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}, \quad \mu^t_{\min} = \frac{1}{4}\frac{b^t_{\min}}{\omega^t_{\max}}. \tag{2.3}$$

Now, we introduce the notion of solution to the system (1)-(5). For $v_0 \in \dot{\mathcal{V}}^2_{\text{div}}$, strictly positive ω_0, $b_0 \in \mathcal{V}^2$ and a positive T, functions $(v, \omega, b) \in \mathcal{X}(T)$ are solution to (1)-(5) if

$$(v_{,t}, w) - (v \otimes v, \nabla w) + (\mu D(v), D(w)) = 0 \quad \text{for} \quad w \in \dot{\mathcal{V}}^1_{\text{div}}, \tag{2.4}$$

$$(\omega_{,t}, z) - (\omega v, \nabla z) + (\mu\nabla\omega, \nabla z) = -\kappa_2(\omega^2, z) \quad \text{for} \quad z \in \mathcal{V}^1, \tag{2.5}$$

$$(b_{,t}, q) - (bv, \nabla q) + (\mu\nabla b, \nabla q) = -(b\omega, q) + (\mu|D(v)|^2, q) \quad \text{for} \quad q \in \mathcal{V}^1 \tag{2.6}$$

for a.a. $t \in (0, T)$, where $\mu = \frac{b}{\omega}$ and (5) holds. We recall that $D(v)$ denotes the symmetric part of ∇v and $(\cdot, \cdot)$ is the inner product in $L^2(\Omega)$.

Our main result concerning the existence of local in-time regular solutions is as follows.

Theorem 2.1.1. *Suppose that ω_0, $b_0 \in \mathcal{V}^2$, $v_0 \in \dot{\mathcal{V}}^2_{\text{div}}$ and (2.1), (2.2) are satisfied. Then there exist positive t^* and $(v, \omega, b) \in \mathcal{X}(t^*)$ such that (2.4)-(2.6) hold for a.a. $t \in (0, t^*)$ and (5) is satisfied. Furthermore, for each $(x, t) \in \Omega \times [0, t^*)$ the following estimates*

$$\frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t} \leqslant \omega(x,t) \leqslant \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}, \tag{2.7}$$

$$\frac{b_{\min}}{(1+\kappa_2\omega_{\max}t)^{\frac{1}{\kappa_2}}} \leqslant b(x,t) \tag{2.8}$$

hold. The time of the existence of the solution is estimated from below in the following sense: for each positive δ and compact $K \subseteq \{(a, b, c) : 0 < a \leqslant b,\ 0 < c\}$ there exists positive $t^*_{K,\delta}$, which depends only on κ_2, Ω, δ and K such that if

$$\|v_0\|^2_{2,2} + \|\omega_0\|^2_{2,2} + \|b_0\|^2_{2,2} \leqslant \delta \quad \textit{and} \quad (\omega_{\min}, \omega_{\max}, b_{\min}) \in K, \tag{2.9}$$

then $t^* \geqslant t^*_{K,\delta}$. The Sobolev norm is defined by (1.1).

We note that the last part of the theorem is needed for proving the existence of a global-in-time solution for small data. We address this issue in Chapter 3.

In the next section, we prove the above theorem by applying the Galerkin method for an appropriate truncated problem. We obtain a priori estimates for the sequence of approximate solutions and by a weak-compactness argument we get a solution of the truncated problem. Finally, after proving some bounds for ω and b we deduce that the obtained solution satisfies the original system of equations.

2.2. Proof of Theorem 2.1.1

The proof of Theorem 2.1.1 is based on the Galerkin method. Hence, we need a basis of the spaces $\mathcal{V}^1$ and $\dot{\mathcal{V}}^1_{\text{div}}$. Let $\{w_i\}_{i\in\mathbb{N}}$ be a system of eigenfunctions of the Stokes operator in $\dot{\mathcal{V}}^1_{\text{div}}$, which is complete and orthogonal in $\dot{\mathcal{V}}^1_{\text{div}}$ and orthonormal in $L^2(\Omega)$ (see Chap. II.6 in [19]). In particular, $\{w_i\}_{i\in\mathbb{N}}$ are smooth (see formula (6.17), Chap. II in [19]). We denote by $\{\lambda_i\}_{i\in\mathbb{N}}$ the corresponding system of eigenvalues. Similarly, let $\{z_i\}_{i\in\mathbb{N}}$ be a complete and orthogonal system in $\mathcal{V}^1$, which is orthonormal in $L^2(\Omega)$. Set $\{z_i\}_{i\in\mathbb{N}}$ is composed of eigenvectors of the minus Laplace operator. The system of corresponding eigenvalues is denoted by $\{\tilde{\lambda}_i\}_{i\in\mathbb{N}}$.We shall find approximate solutions of (2.4)-(2.6) in the following form

$$v^l(t,x) = \sum_{i=1}^{l} c_i^l(t) w_i(x), \quad \omega^l(t,x) = \sum_{i=1}^{l} e_i^l(t) z_i(x), \quad b^l(t,x) = \sum_{i=1}^{l} d_i^l(t) z_i(x). \tag{2.10}$$

We have to determine the coefficients $\{c_i^l\}_{i=1}^l$, $\{e_i^l\}_{i=1}^l$ and $\{d_i^l\}_{i=1}^l$. In order to define an approximate problem we have to introduce a few auxiliary functions. For fixed $t > 0$ we denote by $\Psi_t = \Psi_t(x)$ a smooth function such that

$$\Psi_t(x) = \begin{cases} \frac{1}{2} b^t_{\min} & \text{for} \quad x < \frac{1}{2} b^t_{\min}, \\ x & \text{for} \quad x \geqslant b^t_{\min}, \end{cases} \tag{2.11}$$

where $b^t_{\min}$ is defined by (2.3). We assume that the function Ψ_t also satisfies

$$0 \leqslant \Psi_t'(x) \leqslant c_0, \;\; |\Psi_t''(x)| \leqslant c_0 (b^t_{\min})^{-1}, \tag{2.12}$$

where, c_0 is a constant independent of x and t (see Section 1.4 for details i.e. formula (1.22)). We also introduce the smooth functions Φ_t, ψ_t and ϕ_t:

$$\Phi_t(x) = \begin{cases} \frac{1}{2}\omega^t_{\min} & \text{for} \quad x < \frac{1}{2}\omega^t_{\min}, \\ x & \text{for} \quad x \in [\omega^t_{\min}, \omega^t_{\max}], \\ 2\omega^t_{\max} & \text{for} \quad x > 2\omega^t_{\max}, \end{cases} \tag{2.13}$$

$$\psi_t(x) = \begin{cases} 0 & \text{for} \quad x < \frac{1}{2}b^t_{\min}, \\ x & \text{for} \quad x \geqslant b^t_{\min}, \end{cases} \tag{2.14}$$

$$\phi_t(x) = \begin{cases} 0 & \text{for} \quad x < \frac{1}{2}\omega^t_{\min}, \\ x & \text{for} \quad x \geqslant \omega^t_{\min}. \end{cases} \tag{2.15}$$

We assume that these functions additionally satisfy

$$0 \leqslant \Phi_t'(x) \leqslant c_0, \;\; |\Phi_t''(x)| \leqslant c_0(\omega^t_{\min})^{-1}, \tag{2.16}$$

$$\psi_t(x) \leqslant x \quad \text{for} \quad x \geqslant 0, \quad 0 \leqslant \psi_t'(x) \leqslant c_0 \quad \text{for} \quad x \in \mathbb{R}, \tag{2.17}$$

$$\phi_t(x) \leqslant x \quad \text{for} \quad x \geqslant 0, \quad 0 \leqslant \phi_t'(x) \leqslant c_0 \quad \text{for} \quad x \in \mathbb{R} \tag{2.18}$$

for some constant c_0 (see Section 1.4).

An approximate solution will be found in the form (2.10), where the coefficients $\{c^l_i\}^l_{i=1}$, $\{e^l_i\}^l_{i=1}$ and $\{d^l_i\}^l_{i=1}$ are determined by the following truncated system

$$(v^l_{,t}, w_i) - (v^l \otimes v^l, \nabla w_i) + \big(\mu^l D(v^l), D(w_i)\big) = 0, \tag{2.19}$$

$$(\omega^l_{,t}, z_i) - (\omega^l v^l, \nabla z_i) + \big(\mu^l \nabla \omega^l, \nabla z_i\big) = -\kappa_2(\phi_t^2(\omega^l), z_i), \tag{2.20}$$

$$(b^l_{,t}, z_i) - (b^l v^l, \nabla z_i) + \big(\mu^l \nabla b^l, \nabla z_i\big) = -(\psi_t(b^l)\phi_t(\omega^l), z_i) + (\mu^l |D(v^l)|^2, z_i), \tag{2.21}$$

$$c^l_i(0) = (v_0, w_i), \;\; e^l_i(0) = (\omega_0, z_i), \;\; d^l_i(0) = (b_0, z_i),$$

where $i \in \{1, \ldots, l\}$ and we denote

$$\mu^l = \frac{\Psi_t(b^l)}{\Phi_t(\omega^l)}. \tag{2.22}$$

In the computations below, the exponent l systematically refers to this Galerkin approximation.

Remark 2.2.1. *We emphasise that in order to control the second derivatives of approximated solutions we need the conditions (2.12), (2.16)-(2.18). In particular, we can not apply piecewise linear functions.*

Firstly, we note that μ^l is positive and then by the standard ODE theory the system (2.19)-(2.21) has a local-in-time solution. Now, we shall obtain an estimate independent of l.

Lemma 2.2.2. *The approximate solution obtained above satisfy the following estimates:*

$$\frac{d}{dt}\|v^l\|_2^2 + 2\mu_{\min}^t\|D(v^l)\|_2^2 \leqslant 0, \tag{2.23}$$

$$\frac{d}{dt}\|\omega^l\|_2^2 + 2\mu_{\min}^t\|\nabla\omega^l\|_2^2 \leqslant 0, \tag{2.24}$$

$$\frac{d}{dt}\|b^l\|_2^2 + 2\mu_{\min}^t\|\nabla b^l\|_2^2 \leqslant 2\left\|b^l\right\|_\infty\left\|\mu^l\right\|_\infty\|\nabla v^l\|_2^2, \tag{2.25}$$

where $\mu_{\min}^t$ is defined by (2.3).

Proof. We multiply (2.19) by c_i^l, sum over i to obtain

$$\frac{1}{2}\frac{d}{dt}\|v^l\|_2^2 + (\mu^l D(v^l), D(v^l)) = 0,$$

where we used (2.10) and the fact that $\operatorname{div} v^l = 0$. Applying the properties of functions Ψ_t, Φ_t and (2.3) we get

$$\frac{1}{2}\frac{d}{dt}\|v^l\|_2^2 + \mu_{\min}^t\|D(v^l)\|_2^2 \leqslant 0. \tag{2.26}$$

Similarly, we multiply (2.20) by e_i^l and obtain

$$\frac{1}{2}\frac{d}{dt}\|\omega^l\|_2^2 + (\mu^l\nabla\omega^l, \nabla\omega^l) = -\kappa_2(\phi_t^2(\omega^l), \omega^l).$$

By the properties of ϕ_t the right-hand side is non-positive, thus we obtain (2.24). Finally, after multiplying (2.21) by d_i^l we get

$$\frac{1}{2}\frac{d}{dt}\|b^l\|_2^2 + (\mu^l\nabla b^l, \nabla b^l) = -(\psi_t(b^l)\phi_t(\omega^l), b^l) + (\mu^l|D(v^l)|^2, b^l).$$

We note that $\psi_t(b^l)\phi_t(\omega^l)b^l \geqslant 0$. Hence, we obtain

$$\frac{1}{2}\frac{d}{dt}\|b^l\|_2^2 + \mu_{\min}^t\|\nabla b^l\|_2^2 \leqslant (\mu^l|D(v^l)|^2, b^l) \leqslant \left\|b^l\right\|_\infty \left\|\mu^l\right\|_\infty \|\nabla v^l\|_2^2$$

and the proof is finished. □

We also need the higher-order estimates.

Lemma 2.2.3. *There exist positive t^* and C_*, which depend on $b_{\min}$, $\omega_{\min}$, $\omega_{\max}$, Ω, κ_2, c_0, $\|v_0\|_{2,2}$, $\|\omega_0\|_{2,2}$ and $\|b_0\|_{2,2}$ such that for each $l \in \mathbb{N}$ the following estimate*

$$\|v^l, \omega^l, b^l\|_{L^\infty(0,t^*;H^2(\Omega))} + \|v^l, \omega^l, b^l\|_{L^2(0,t^*;H^3(\Omega))} + \|v^l_{,t}, \omega^l_{,t}, b^l_{,t}\|_{L^2(0,t^*;H^1(\Omega))} \leqslant C_* \tag{2.27}$$

holds.

Furthermore, for each positive δ and compact $K \subseteq \{(a,b,c) : 0 < a \leqslant b,\ \ 0 < c\}$ there exists a positive $t^*_{K,\delta}$, which depends only on κ_2, Ω, δ and K such that if

$$\|v_0\|_{2,2}^2 + \|\omega_0\|_{2,2}^2 + \|b_0\|_{2,2}^2 \leqslant \delta \quad \text{and} \quad (\omega_{\min}, \omega_{\max}, b_{\min}) \in K,$$

then $t^* \geqslant t^*_{K,\delta}$.

Before we pass to the proof of Lemma 2.2.3 we present its idea. First, we test the equation for an approximate solution by its bi-Laplacian. Next, after integration by parts we obtain (2.28), (2.29) and (2.30). Further, we apply the lower bound for the "diffusive coefficient" μ^l (see (2.33)) and use the Hölder and Gagliardo-Nirenberg inequalities which leads to (2.46). To estimate the H^2-norm of μ^l we use the properties of Ψ_t and Φ_t. After applying the energy estimates from Lemma 2.2.2 we obtain (2.57), which leads to a uniform bound of the H^2-norm of the sequence of approximate solutions on the interval $(0, t^*)$ for some positive t^* (see (2.61)). Immediately it gives a bound in $L^2(0, T, H^3(\Omega))$. The last step is an l-independent estimate of the time derivative of the approximate solution.

Proof. We multiply the equality (2.19) by $\lambda_i^2 c_i^l$ and sum over i

$$(v^l_{,t}, \Delta^2 v^l) - (v^l \otimes v^l, \nabla\Delta^2 v^l) + (\mu^l D(v^l), D(\Delta^2 v^l)) = 0.$$

After integrating by parts we obtain

$$(v^l_{,t}, \Delta^2 v^l) = \frac{1}{2}\frac{d}{dt}\|\Delta v^l\|_2^2,$$

$$(v^l \otimes v^l, \nabla\Delta^2 v^l) = (\Delta(v^l \otimes v^l), \nabla\Delta v^l),$$

$$\begin{aligned}(\mu^l D(v^l), D(\Delta^2 v^l)) &= (\Delta\mu^l D(v^l), \Delta D(v^l)) + 2(\nabla\mu^l \cdot \nabla D(v^l), \Delta D(v^l)) \\ &+ (\mu^l \Delta D(v^l), \Delta D(v^l)).\end{aligned}$$

Thus, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta v^l\|_2^2 + \int_\Omega \mu^l |\Delta D(v^l)|^2 dx &= (\Delta(v^l \otimes v^l), \nabla\Delta v^l) - (\Delta\mu^l D(v^l), \Delta D(v^l)) \\ &- 2(\nabla\mu^l \cdot \nabla D(v^l), \Delta D(v^l)).\end{aligned}$$

We estimate the right-hand side

$$|(\Delta(v^l \otimes v^l), \nabla\Delta v^l)| \leqslant C(\left\|v^l\right\|_\infty \left\|\nabla^2 v^l\right\|_2 \left\|\nabla^3 v^l\right\|_2 + \|\nabla v^l\|_4^2 \left\|\nabla^3 v^l\right\|_2).$$

Proceeding analogously we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta v^l\|_2^2 + \int_\Omega \mu^l |\Delta D(v^l)|^2 dx &\leqslant C(\left\|v^l\right\|_\infty \left\|\nabla^2 v^l\right\|_2 \left\|\nabla^3 v^l\right\|_2 + \|\nabla v^l\|_4^2 \left\|\nabla^3 v^l\right\|_2) \\ &+ \left(\left\|\Delta\mu^l D(v^l)\right\|_2 + 2\left\|\nabla\mu^l \cdot \nabla D(v^l)\right\|_2 \right) \left\|\Delta D(v^l)\right\|_2.\end{aligned} \tag{2.28}$$

Now, we multiply the equation (2.20) by $\tilde{\lambda}_i^2 e_i^l$ and obtain

$$(\omega^l_{,t}, \Delta^2\omega^l) - (\omega^l v^l, \nabla\Delta^2\omega^l) + \left(\mu^l \nabla\omega^l, \nabla\Delta^2\omega^l\right) = -\kappa_2(\phi_t^2(\omega^l), \Delta^2\omega^l).$$

After integrating by parts we get

$$(\omega^l_{,t}, \Delta^2\omega^l) = \frac{1}{2}\frac{d}{dt}\|\Delta\omega^l\|_2^2,$$

$$(\omega^l v^l, \nabla\Delta^2\omega^l) = (\Delta\omega^l v^l, \nabla\Delta\omega^l) + 2(\nabla v^l \nabla\omega^l, \nabla\Delta\omega^l) + (\omega^l \Delta v^l, \nabla\Delta\omega^l),$$

$$\left(\mu^l \nabla\omega^l, \nabla\Delta^2\omega^l\right) = \left(\Delta\mu^l \nabla\omega^l, \nabla\Delta\omega^l\right) + 2\left(\nabla^2\omega^l \nabla\mu^l, \nabla\Delta\omega^l\right) + \left(\mu^l \nabla\Delta\omega^l, \nabla\Delta\omega^l\right),$$

$$-(\phi_t^2(\omega^l), \Delta^2\omega^l) = 2\left(\phi_t(\omega^l)\phi_t'(\omega^l)\nabla\omega^l, \nabla\Delta\omega^l\right).$$

Thus, we may write

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Delta\omega^l\|_2^2 + \int_\Omega \mu^l \left|\nabla\Delta\omega^l\right|^2 dx \leqslant \big(&\left\|\Delta\omega^l v^l\right\|_2 + 2\left\|\nabla v^l \nabla\omega^l\right\|_2 + \left\|\omega^l \Delta v^l\right\|_2 \\
&+ \left\|\Delta\mu^l \nabla\omega^l\right\|_2 + 2\left\|\nabla^2\omega^l \nabla\mu^l\right\|_2 + 2\kappa_2 \left\|\phi_t(\omega^l)\phi_t'(\omega^l)\nabla\omega^l\right\|_2\big) \left\|\nabla\Delta\omega^l\right\|_2.
\end{aligned} \tag{2.29}
$$

Finally, after multiplying (2.21) by $\tilde{\lambda}_i^2 d_i^l$ we obtain

$$
(b^l_{,t}, \Delta^2 b^l) - (b^l v^l, \nabla\Delta^2 b^l) + \left(\mu^l \nabla b^l, \nabla\Delta^2 b^l\right) = -(\psi_t(b^l)\phi_t(\omega^l), \Delta^2 b^l) + (\mu^l |D(v^l)|^2, \Delta^2 b^l).
$$

We deal with the terms on the left-hand side as earlier and for the right-hand side terms we get

$$
-(\psi_t(b^l)\phi_t(\omega^l), \Delta^2 b^l) = \left(\psi_t'(b^l)\phi_t(\omega^l)\nabla b^l, \nabla\Delta b^l\right) + \left(\psi_t(b^l)\phi_t'(\omega^l)\nabla\omega^l, \nabla\Delta b^l\right),
$$

$$
(\mu^l |D(v^l)|^2, \Delta^2 b^l) = -(|D(v^l)|^2 \nabla\mu^l, \nabla\Delta b^l) - (\mu^l \nabla(|D(v^l)|^2), \nabla\Delta b^l).
$$

Therefore, we obtain the inequality

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Delta b^l\|_2^2 + \int_\Omega \mu^l \left|\nabla\Delta b^l\right|^2 dx \leqslant \Big(&\left\|\Delta b^l v^l\right\|_2 + 2\left\|\nabla v^l \nabla b^l\right\|_2 + \left\|b^l \Delta v^l\right\|_2 \\
&+ \left\|\Delta\mu^l \nabla b^l\right\|_2 + 2\left\|\nabla^2 b^l \nabla\mu^l\right\|_2 + \left\|\phi_t(\omega^l)\psi_t'(b^l)\nabla b^l\right\|_2 \\
&+ \left\|\psi_t(b^l)\phi_t'(\omega^l)\nabla\omega^l\right\|_2 + \left\|\nabla\mu^l \left|D(v^l)\right|^2\right\|_2 + 2\left\|\mu^l |D(v^l)||\nabla D(v^l)|\right\|_2\Big) \left\|\nabla\Delta b^l\right\|_2.
\end{aligned} \tag{2.30}
$$

We note that

$$
\int_\Omega \left|\Delta D(v^l)\right|^2 dx = \frac{1}{2}\int_\Omega \left|\nabla^3 v^l\right|^2 dx. \tag{2.31}
$$

Indeed, integration by parts yields

$$
\begin{aligned}
2\int_\Omega \left|\Delta D(v^l)\right|^2 dx &= \sum_{k,m}\int_\Omega \left|\Delta v^l_{k,x_m}\right|^2 dx + \int_\Omega \Delta v^l_{k,x_m} \cdot \Delta v^l_{m,x_k} dx \\
&= \sum_{k,m,p,q}\int_\Omega v^l_{k,x_m x_p x_p} \cdot v^l_{k,x_m x_q x_q} dx + \sum_{k,m,p,q}\int_\Omega \Delta v^l_{k,x_k} \cdot \Delta v^l_{m,x_m} dx \\
&= \sum_{k,m,p,q}\int_\Omega \left|v^l_{k,x_m x_p x_q}\right|^2 dx,
\end{aligned}
$$

where we applied the condition $\operatorname{div} v^l = 0$ and used the tensor notation for components and derivatives. Also by the same argument the following holds

$$\|\Delta v^l\|_2^2 = \|\nabla^2 v^l\|_2^2. \tag{2.32}$$

After applying (2.3), (2.11), (2.13) and (2.22) we get

$$\mu_{\min}^t \leqslant \mu^l \tag{2.33}$$

for each l. Thus, (2.28) together with (2.31) and (2.33) give

$$\begin{aligned} \frac{d}{dt}\|\Delta v^l\|_2^2 + \mu_{\min}^t \|\Delta D(v^l)\|_2^2 \leqslant \frac{C}{\mu_{\min}^t}\Big(&\|v^l\|_\infty^2 \|\nabla^2 v^l\|_2^2 + \|\nabla v^l\|_4^4 + \|\Delta \mu^l D(v^l)\|_2^2 \\ &+ \|\nabla \mu^l \cdot \nabla D(v^l)\|_2^2\Big). \end{aligned} \tag{2.34}$$

Applying the Sobolev embedding inequality and the Gagliardo-Nirenberg interpolation inequality

$$\left\|\nabla v^l\right\|_\infty \leqslant C_1 \left\|\nabla^2 v^l\right\|_4 \leqslant C \left\|\nabla^3 v^l\right\|_2^{\frac{3}{4}} \|\nabla v^l\|_6^{\frac{1}{4}} \tag{2.35}$$

we get

$$\|\Delta \mu^l D(v^l)\|_2^2 \leqslant \|\Delta \mu^l\|_2^2 \|D(v^l)\|_\infty^2 \leqslant C \left\|\nabla^3 v^l\right\|_2^{\frac{3}{2}} \|v^l\|_{2,2}^{\frac{1}{2}} \|\mu^l\|_{2,2}^2,$$

where C depends only on Ω. Thus, applying the Young inequality with exponents $(\frac{4}{3}, 20, 5)$ we get

$$\|\Delta \mu^l D(v^l)\|_2^2 \leqslant \varepsilon \|\nabla^3 v^l\|_2^2 + \frac{C}{\varepsilon^3}(\|v^l\|_{2,2}^{10} + \|\mu^l\|_{2,2}^{10}), \tag{2.36}$$

Again, by the Gagliardo-Nirenberg inequality

$$\|\nabla^2 v^l\|_3 \leqslant C \left\|\nabla^3 v^l\right\|_2^{\frac{1}{2}} \left\|\nabla^2 v^l\right\|_2^{\frac{1}{2}} \tag{2.37}$$

and the Hölder inequality we have

$$\|\nabla \mu^l \cdot \nabla D(v^l)\|_2^2 \leqslant \|\nabla \mu\|_6^2 \|\nabla^2 v^l\|_3^2 \leqslant C \left\|\nabla^3 v^l\right\|_2 \|v^l\|_{2,2} \|\mu^l\|_{2,2}^2.$$

Thus, applying the Young inequality with exponents $(2, 6, 3)$ we get

$$\|\nabla \mu^l \cdot \nabla D(v^l)\|_2^2 \leqslant \varepsilon \|\nabla^3 v^l\|_2^2 + \frac{C}{\varepsilon}(\|v^l\|_{2,2}^6 + \|\mu^l\|_{2,2}^6), \tag{2.38}$$

where $\varepsilon > 0$ and C depends only on Ω. Applying the above inequalities and (2.31), (2.32) in (2.34) we obtain

$$\begin{aligned}\frac{d}{dt}\|\nabla^2 v^l\|_2^2 + \frac{\mu^t_{\min}}{4}\|\nabla^3 v^l\|_2^2 \leqslant & \frac{C}{\mu^t_{\min}}\Big(\|v^l\|^4_{2,2} + (\mu^t_{\min})^{-2}(\|v^l\|^6_{2,2} + \|\mu^l\|^6_{2,2}) \\ & + (\mu^t_{\min})^{-6}(\|v^l\|^{10}_{2,2} + \|\mu^l\|^{10}_{2,2})\Big),\end{aligned} \tag{2.39}$$

where $C = C(\Omega)$. Now, we proceed similarly with (2.29) and we obtain

$$\begin{aligned}\frac{d}{dt}\|\Delta\omega^l\|_2^2 + \mu^t_{\min}\|\nabla\Delta\omega^l\|_2^2 \leqslant & \frac{C}{\mu^t_{\min}}\Big(\|v^l\|^2_\infty\|\nabla^2\omega^l\|^2_2 + \|\nabla v^l\|^2_4\|\nabla\omega^l\|^2_4 + \|\omega^l\|^2_\infty\|\nabla^2 v^l\|^2_2 \\ & + \|\Delta\mu^l\nabla\omega^l\|^2_2 + \|\nabla^2\omega^l\nabla\mu^l\|^2_2 + \kappa_2^2 c_0^2\|\omega^l\|^2_\infty\|\nabla\omega^l\|^2_2\Big),\end{aligned} \tag{2.40}$$

where we applied (2.18). We repeat the reasoning, which leads to (2.36), (2.38) and get

$$\begin{aligned}\|\Delta\mu^l\nabla\omega^l\|^2_2 &\leqslant \varepsilon\|\nabla^3\omega^l\|^2_2 + \frac{C}{\varepsilon^3}(\|\omega^l\|^{10}_{2,2} + \|\mu^l\|^{10}_{2,2}),\\ \|\nabla^2\omega^l\nabla\mu^l\|^2_2 &\leqslant \varepsilon\|\nabla^3\omega^l\|^2_2 + \frac{C}{\varepsilon}(\|\omega^l\|^6_{2,2} + \|\mu^l\|^6_{2,2}).\end{aligned}$$

Thus, the above inequalities and (2.40) give

$$\begin{aligned}\frac{d}{dt}\|\nabla^2\omega^l\|_2^2 + \frac{\mu^t_{\min}}{2}\|\nabla^3\omega^l\|_2^2 \leqslant & \frac{C}{\mu^t_{\min}}\Big(\|v^l\|^4_{2,2} + (1+\kappa_2^4 c_0^4)\|\omega^l\|^4_{2,2} \\ & + (\mu^t_{\min})^{-2}(\|\omega^l\|^6_{2,2} + \|\mu^l\|^6_{2,2}) + (\mu^t_{\min})^{-6}(\|\omega^l\|^{10}_{2,2} + \|\mu^l\|^{10}_{2,2})\Big),\end{aligned} \tag{2.41}$$

where $C = C(\Omega)$. Further, from (2.30) we obtain

$$\begin{aligned}\frac{d}{dt}\|\Delta b^l\|_2^2 + \mu^t_{\min}\|\nabla\Delta b^l\|_2^2 \leqslant & \frac{C}{\mu^t_{\min}}\Big(\|v^l\|^2_\infty\|\nabla^2 b^l\|^2_2 + \|\nabla v^l\|^2_4\|\nabla b^l\|^2_4 + \|b^l\|^2_\infty\|\nabla^2 v^l\|^2_2 \\ & + \|\Delta\mu^l\nabla b^l\|^2_2 + \|\nabla^2 b^l\nabla\mu^l\|^2_2 + c_0^2\|\omega^l\|^2_\infty\|\nabla b^l\|^2_2 \\ & + c_0^2\|b^l\|^2_\infty\|\nabla\omega^l\|^2_2 + \|\nabla\mu^l|D(v^l)|^2\|^2_2 + \|\mu^l|D(v^l)||\nabla D(v^l)|\|^2_2\Big),\end{aligned}$$

where we used (2.17) and (2.18). Applying the Hölder inequality, the Young inequality and the Sobolev embedding theorem we get

$$\begin{aligned}\frac{d}{dt}\|\nabla^2 b^l\|_2^2 + \mu^t_{\min}\|\nabla^3 b^l\|_2^2 \leqslant & \frac{C}{\mu^t_{\min}}\Big(\|v^l\|^4_{2,2} + \|b^l\|^4_{2,2} + \|\Delta\mu^l\nabla b^l\|^2_2 + \|\nabla^2 b^l\nabla\mu^l\|^2_2 \\ & + c_0^4\|\omega^l\|^4_{2,2} + \|\mu^l\|^6_{2,2} + \|v^l\|^6_{2,2} + \|\nabla^2 v^l\|^2_3\|\mu^l\|^2_{2,2}\|v^l\|^2_{2,2}\Big).\end{aligned} \tag{2.42}$$

Applying again the Gagliardo-Nirenberg inequality and the Young inequality we get

$$\begin{aligned}
\|\Delta\mu^l\nabla b^l\|_2^2 &\leqslant \varepsilon\|\nabla^3 b^l\|_2^2 + \frac{C}{\varepsilon^3}(\|b^l\|_{2,2}^{10} + \|\mu^l\|_{2,2}^{10}),\\
\|\nabla^2 b^l\nabla\mu^l\|_2^2 &\leqslant \varepsilon\|\nabla^3 b^l\|_2^2 + \frac{C}{\varepsilon}(\|b^l\|_{2,2}^{6} + \|\mu^l\|_{2,2}^{6}).
\end{aligned}$$

From (2.37) we obtain

$$\|\nabla^2 v^l\|_3^2\|v^l\|_{2,2}^2\|\mu^l\|_{2,2}^2 \leqslant C\left\|\nabla^3 v^l\right\|_2\|v^l\|_{2,2}^3\|\mu^l\|_{2,2}^2 \leqslant \varepsilon\|\nabla^3 v^l\|_2^2 + \frac{C}{\varepsilon}(\|v^l\|_{2,2}^{10} + \|\mu^l\|_{2,2}^{10}).$$

Based on (2.42) we deduce the following estimate

$$\begin{aligned}
\frac{d}{dt}\|\nabla^2 b^l\|_2^2 + \frac{\mu_{\min}^t}{2}\|\nabla^3 b^l\|_2^2 \leqslant &\frac{C}{\mu_{\min}^t}\Big(\|v^l\|_{2,2}^4 + \|b^l\|_{2,2}^4 + c_0^4\|\omega^l\|_{2,2}^4 + \|\mu^l\|_{2,2}^6 + \|v^l\|_{2,2}^6\Big)\\
&+ \frac{C}{(\mu_{\min}^t)^3}\Big(\|b^l\|_{2,2}^6 + \|\mu^l\|_{2,2}^6 + \|v^l\|_{2,2}^{10} + \|\mu^l\|_{2,2}^{10}\Big)\\
&+ \frac{C}{(\mu_{\min}^t)^7}\Big(\|b^l\|_{2,2}^{10} + \|\mu^l\|_{2,2}^{10}\Big) + \frac{\mu_{\min}^t}{8}\|\nabla^3 v^l\|_2^2,
\end{aligned} \tag{2.43}$$

where $C = C(\Omega)$. We sum the inequalities (2.39), (2.41), (2.43) and we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(\|\nabla^2 v^l\|_2^2 + \|\nabla^2\omega^l\|_2^2 + \|\nabla^2 b^l\|_2^2\Big) + \frac{\mu_{\min}^t}{8}\Big(\|\nabla^3 v^l\|_2^2 + \|\nabla^3\omega^l\|_2^2 + \|\nabla^3 b^l\|_2^2\Big)\\
&\qquad\leqslant \frac{C}{\mu_{\min}^t}\Big(\|v^l\|_{2,2}^4 + \|b^l\|_{2,2}^4 + (1 + c_0^4 + c_0^4\kappa_2^4)\|\omega^l\|_{2,2}^4 + \|\mu^l\|_{2,2}^6 + \|v^l\|_{2,2}^6\Big)\\
&\qquad+ \frac{C}{(\mu_{\min}^t)^3}\Big(\|v^l\|_{2,2}^6 + \|b^l\|_{2,2}^6 + \|\omega^l\|_{2,2}^6 + \|\mu^l\|_{2,2}^6 + \|v^l\|_{2,2}^{10} + \|\mu^l\|_{2,2}^{10}\Big)\\
&\qquad+ \frac{C}{(\mu_{\min}^t)^7}\Big(\|v^l\|_{2,2}^{10} + \|\omega^l\|_{2,2}^{10} + \|b^l\|_{2,2}^{10} + \|\mu^l\|_{2,2}^{10}\Big)
\end{aligned} \tag{2.44}$$

for some C, which depends only on Ω. We note that

$$\mu_{\min}^t = \frac{1}{4}\frac{b_{\min}}{\omega_{\max}}(1 + \kappa_2\omega_{\max}t)^{1-\frac{1}{\kappa_2}}. \tag{2.45}$$

Hence, we have

$$\begin{aligned}
&\frac{d}{dt}\Big(\|\nabla^2 v^l\|_2^2 + \|\nabla^2\omega^l\|_2^2 + \|\nabla^2 b^l\|_2^2\Big) + \frac{\mu_{\min}^t}{8}\Big(\|\nabla^3 v^l\|_2^2 + \|\nabla^3\omega^l\|_2^2 + \|\nabla^3 b^l\|_2^2\Big)\\
&\quad\leqslant CK(b_{\min},\omega_{\max})\left(1 + \kappa_2\omega_{\max}t\right)^{\beta}\Big(1 + \|b^l\|_{2,2}^{10} + \|\omega^l\|_{2,2}^{10} + \|\mu^l\|_{2,2}^{10} + \|v^l\|_{2,2}^{10}\Big),
\end{aligned} \tag{2.46}$$

where $K(b_{\min}, \omega_{\max}) = \frac{\omega_{\max}}{b_{\min}} + \left(\frac{\omega_{\max}}{b_{\min}}\right)^3 + \left(\frac{\omega_{\max}}{b_{\min}}\right)^7$, $\beta = \max\{\frac{1}{\kappa_2} - 1, \frac{3}{\kappa_2} - 3, \frac{7}{\kappa_2} - 7\}$ and C depends only on Ω, c_0 and κ_2.

Now, we shall estimate μ^l in terms of ω^l and b^l. Firstly, we note that from (2.11) and (2.13) we have

$$\Psi_t(b^l) \leqslant \frac{1}{2} b^t_{\min} + |b^l|, \quad \Phi_t(\omega^l) \geqslant \frac{1}{2}\omega^t_{\min}. \tag{2.47}$$

Hence, by definition (2.22) we get

$$0 < \mu^l \leqslant 2(\omega^t_{\min})^{-1}\left(\frac{1}{2}b_{\min} + |b^l|\right) = \frac{1}{\omega_{\min}}\left(1 + \kappa_2\omega_{\min}t\right)\left(b_{\min} + 2|b^l|\right). \tag{2.48}$$

Thus, we obtain

$$\left\|\mu^l\right\|_2 \leqslant \frac{c_1}{\omega_{\min}}\left(1 + \kappa_2\omega_{\min}t\right)\left(b_{\min} + \left\|b^l\right\|_2\right), \tag{2.49}$$

where c_1 depends only on Ω. Now, we have to estimate the derivatives of μ^l. Direct calculation gives

$$\begin{aligned}|\nabla^2\mu^l| &= \left|\nabla^2\left(\Psi_t(b^l)\cdot(\Phi_t(\omega^l))^{-1}\right)\right| \\ &\leqslant (\Phi_t(\omega^l))^{-1}\left|\nabla^2(\Psi_t(b^l))\right| + 2(\Phi_t(\omega^l))^{-2}\left|\nabla(\Psi_t(b^l))\right|\left|\nabla(\Phi_t(\omega^l))\right| \\ &\quad + 2\Psi_t(b^l)(\Phi_t(\omega^l))^{-3}\left|\nabla(\Phi_t(\omega^l))\right|^2 + \Psi_t(b^l)(\Phi_t(\omega^l))^{-2}\left|\nabla^2(\Phi_t(\omega^l))\right|.\end{aligned} \tag{2.50}$$

Using (2.12) and (2.16) we may estimate the derivatives

$$\left|\nabla(\Psi_t(b^l))\right| \leqslant c_0\left|\nabla b^l\right|, \quad \left|\nabla(\Phi_t(\omega^l))\right| \leqslant c_0\left|\nabla\omega^l\right|, \tag{2.51}$$

$$\begin{aligned}\left|\nabla^2(\Psi_t(b^l))\right| &\leqslant c_0(b^t_{\min})^{-1}\left|\nabla b^l\right|^2 + c_0\left|\nabla^2 b^l\right|, \\ \left|\nabla^2(\Phi_t(\omega^l))\right| &\leqslant c_0(\omega^t_{\min})^{-1}\left|\nabla\omega^l\right|^2 + c_0\left|\nabla^2\omega^l\right|.\end{aligned} \tag{2.52}$$

If we apply estimates (2.47), (2.51) and (2.52) in (2.50) then we obtain

$$\begin{aligned}\left|\nabla^2\mu^l\right| \leqslant c_2 Q_1\left(1 + \kappa_2\omega_{\max}t\right)^{\max\{3, 1+\frac{1}{\kappa_2}\}}\Big[&\left|\nabla b^l\right|^2 + \left|\nabla^2 b^l\right| + |b^l|\left|\nabla\omega^l\right|^2 \\ &+ \left|\nabla b^l\right| + \left|\nabla\omega^l\right| + \left|\nabla\omega^l\right|^2 + \left|b^l\nabla^2\omega^l\right| + \left|\nabla^2\omega^l\right|\Big],\end{aligned} \tag{2.53}$$

where c_2 depends only on c_0 and $Q_1 = \frac{b_{\min}}{\omega_{\min}}\left(1 + b_{\min}^{-3} + \omega_{\min}^{-3}\right)$. Thus we get

$$\begin{aligned} \left\|\nabla^2\mu^l\right\|_2 \leqslant c_2 Q_1 \left(1+\kappa_2\omega_{\max}t\right)^{\max\{3,1+\frac{1}{\kappa_2}\}} \Big[\|\nabla b^l\|_4^2 + \left\|\nabla^2 b^l\right\|_2 \\ + \left\|b^l\right\|_\infty \|\nabla\omega^l\|_4^2 + \|\nabla\omega^l\|_4^2 + \left\|\nabla^2\omega^l\right\|_2 + \left\|b^l\right\|_\infty \left\|\nabla^2\omega^l\right\|_2\Big]. \end{aligned} \tag{2.54}$$

If we take into account (2.49) then we get

$$\|\mu^l\|_{2,2} \leqslant c_3 Q_1 \left(1+\kappa_2\omega_{\max}t\right)^{\max\{3,1+\frac{1}{\kappa_2}\}} \left(\|b^l\|_{2,2}^3 + \|\omega^l\|_{2,2}^3 + 1\right), \tag{2.55}$$

where $c_3 = c_3(c_0, \Omega)$. Applying the above estimate in (2.46) we obtain

$$\begin{aligned} \frac{d}{dt}\Big(\|\nabla^2 v^l\|_2^2 + \|\nabla^2\omega^l\|_2^2 + \|\nabla^2 b^l\|_2^2\Big) + \frac{\mu_{\min}^t}{8}\Big(\|\nabla^3 v^l\|_2^2 + \|\nabla^3\omega^l\|_2^2 + \|\nabla^3 b^l\|_2^2\Big) \\ \leqslant CQ_2\left(1+\kappa_2\omega_{\max}t\right)^{\bar\beta}\Big(1 + \|v^l\|_{2,2}^2 + \|b^l\|_{2,2}^2 + \|\omega^l\|_{2,2}^2\Big)^{15}, \end{aligned} \tag{2.56}$$

where

$$Q_2 = \left[1 + \left(\frac{\omega_{\max}}{b_{\min}}\right)^7\right]\left[\left(\frac{b_{\min}}{\omega_{\min}}(1 + b_{\min}^{-3} + \omega_{\min}^{-3})\right)^{10} + 1\right],\ \ \bar\beta = 10\max\{1 + \frac{1}{\kappa_2}, 3\} + \beta$$

and C depends only on Ω, c_0 and κ_2. If we take into account the estimates (2.23)-(2.25) then we have

$$\begin{aligned} \frac{d}{dt}\Big(\|v^l\|_{2,2}^2 + \|\omega^l\|_{2,2}^2 + \|b^l\|_{2,2}^2\Big) + \frac{\mu_{\min}^t}{8}\Big(\|v^l\|_{3,2}^2 + \|\omega^l\|_{3,2}^2 + \|b^l\|_{3,2}^2\Big) \\ \leqslant CQ_3\left(1+\kappa_2\omega_{\max}t\right)^{\bar\beta}\Big(1 + \|v^l\|_{2,2}^2 + \|b^l\|_{2,2}^2 + \|\omega^l\|_{2,2}^2\Big)^{15}, \end{aligned} \tag{2.57}$$

where $C = C(c_0, \Omega, \kappa_2)$ and $Q_3 = Q_1^2 + Q_2 + 1$. If we divide both sides by the last term and next integrate with respect to the time variable, we get

$$\begin{aligned} &\Big(1 + \|v^l(t)\|_{2,2}^2 + \|b^l(t)\|_{2,2}^2 + \|\omega^l(t)\|_{2,2}^2\Big)^{-14} \\ &\geqslant \Big(1 + \|v^l(0)\|_{2,2}^2 + \|b^l(0)\|_{2,2}^2 + \|\omega^l(0)\|_{2,2}^2\Big)^{-14} - \frac{14CQ_3}{(\bar\beta+1)\kappa_2\omega_{\max}}\Big((1+\kappa_2\omega_{\max}t)^{\bar\beta+1} - 1\Big) \\ &\geqslant \Big(1 + \|v_0\|_{2,2}^2 + \|b_0\|_{2,2}^2 + \|\omega_0\|_{2,2}^2\Big)^{-14} - \frac{14CQ_3}{(\bar\beta+1)\kappa_2\omega_{\max}}\Big((1+\kappa_2\omega_{\max}t)^{\bar\beta+1} - 1\Big), \end{aligned} \tag{2.58}$$

where the last estimate is a consequence of the Bessel inequality. Now, we define time t^* as the unique solution of the equality

$$\left(1 + \|v_0\|_{2,2}^2 + \|b_0\|_{2,2}^2 + \|\omega_0\|_{2,2}^2\right)^{-14} = \frac{15CQ_3}{(\bar{\beta}+1)\kappa_2\omega_{\max}} \left((1+\kappa_2\omega_{\max}t^*)^{\bar{\beta}+1} - 1\right). \quad (2.59)$$

We note that t^* is positive and depends on $\|v_0\|_{2,2}^2 + \|b_0\|_{2,2}^2 + \|\omega_0\|_{2,2}^2$, κ_2, Ω, c_0, $\omega_{\min}$, $\omega_{\max}$ and $b_{\min}$. It is evident that t^* is decreasing function of $\|v_0\|_{2,2}^2 + \|b_0\|_{2,2}^2 + \|\omega_0\|_{2,2}^2$. Moreover, for any $\delta > 0$ and compact $K \subseteq \{(a,b,c) : \ 0 < a \leqslant b, \ 0 < c\}$ there exists $t^*_{K,\delta} > 0$ such that $t^* \geqslant t^*_{K,\delta}$ for any initial data satisfying $\|v_0\|_{2,2}^2 + \|b_0\|_{2,2}^2 + \|\omega_0\|_{2,2}^2 \leqslant \delta$ and $(\omega_{\min}, \omega_{\max}, b_{\min}) \in K$. From (2.59) we deduce that $t^*_{K,\delta}$ depends only on δ, K, Ω κ_2 and c_0.

From (2.58), (2.59) and the fact that $\bar{\beta}+1 > 0$ we have

$$\left(1 + \|v^l(t)\|_{2,2}^2 + \|b^l(t)\|_{2,2}^2 + \|\omega^l(t)\|_{2,2}^2\right)^{-14} \geqslant \frac{CQ_3}{(\bar{\beta}+1)\kappa_2\omega_{\max}} \left((1+\kappa_2\omega_{\max}t^*)^{\bar{\beta}+1} - 1\right)$$

for $t \in [0,t^*]$. Hence,

$$\|v^l(t)\|_{2,2}^2 + \|b^l(t)\|_{2,2}^2 + \|\omega^l(t)\|_{2,2}^2 \leqslant \left[\frac{CQ_3}{(\bar{\beta}+1)\kappa_2\omega_{\max}} \left((1+\kappa_2\omega_{\max}t^*)^{\bar{\beta}+1} - 1\right)\right]^{-\frac{1}{14}} \quad (2.60)$$

for $t \in [0,t^*]$. In particular, there exists $C^* = C^*(t^*)$ such that

$$\|v^l\|_{L^\infty(0,t^*;\dot{\mathcal{V}}^2_{\mathrm{div}})} + \|\omega^l\|_{L^\infty(0,t^*;\mathcal{V}^2)} + \|b^l\|_{L^\infty(0,t^*;\mathcal{V}^2)} \leqslant C^* \quad (2.61)$$

uniformly with respect to $l \in \mathbb{N}$. Next, from (2.45), (2.57) and (2.61) the bound follows

$$\|v^l\|_{L^2(0,t^*;\dot{\mathcal{V}}^3_{\mathrm{div}})} + \|\omega^l\|_{L^2(0,t^*;\mathcal{V}^3)} + \|b^l\|_{L^2(0,t^*;\mathcal{V}^3)} \leqslant C_*, \quad (2.62)$$

where C_* depends on t^*, κ_2, $b_{\min}$, $\omega_{\max}$ and C^*. It remains to show the estimate of the time derivative of the solution. We do this by multiplying the equality (2.19) by $\frac{d}{dt}c_i^l$ and after summing it over i we get

$$(v^l_{,t}, v^l_{,t}) - (v^l \otimes v^l, \nabla v^l_{,t}) + (\mu^l D(v^l), D(v^l_{,t})) = 0.$$

Thus, after integration by parts and applying the Hölder inequality, we get

$$\|v^l_{,t}\|_2^2 \leqslant \left\|\operatorname{div}(v^l \otimes v^l)\right\|_2 \left\|v^l_{,t}\right\|_2 + \left\|\nabla\left(\mu^l D(v^l)\right)\right\|_2 \left\|v^l_{,t}\right\|_2 .$$

By applying the Young inequality we get

$$\|v^l_{,t}\|_2^2 \leqslant C(\|\operatorname{div}(v^l \otimes v^l)\|_2^2 + \|\nabla\left(\mu^l D(v^l)\right)\|_2^2).$$

Next, the Hölder inequality gives us

$$\|v^l_{,t}\|_2^2 \leqslant C\Big(\|\nabla v^l\|_4^2\|v^l\|_4^2 + \|\nabla\mu^l\|_4^2\|D(v^l)\|_4^2 + \left\|\mu^l\right\|_\infty^2 \|\nabla D(v^l)\|_2^2\Big).$$

Finally, the Sobolev embedding theorem leads us to the following inequality

$$\|v^l_{,t}\|_2^2 \leqslant C\Big(\|v^l\|_{2,2}^4 + \|\mu^l\|_{2,2}^2\|v^l\|_{2,2}^2\Big),$$

where C depends only on Ω. Applying (2.55) and (2.61) we get

$$\|v^l_{,t}\|_{L^\infty(0,t*;L^2(\Omega))} \leqslant C_*, \tag{2.63}$$

where C_* depends on $\Omega, c_0, t^*, \kappa_2$, $b_{\min}$, $\omega_{\max}$ and C^*.

Now, we shall consider (2.20). Proceeding as before we get

$$\begin{aligned}\|\omega^l_{,t}\|_2^2 &\leqslant C(\|\nabla\omega^l \cdot v^l\|_2^2 + \|\nabla(\mu^l\nabla\omega^l)\|_2^2 + \kappa_2\|\phi_t^2(\omega^l)\|_2^2)\\ &\leqslant C(\|v^l\|_\infty^2\|\nabla\omega^l\|_2^2 + \|\nabla\mu^l\|_4^2\|\nabla\omega^l\|_4^2 + \|\mu^l\|_\infty^2\|\nabla^2\omega^l\|_2^2 + \kappa_2\|\omega^l\|_4^4),\end{aligned}$$

where we applied (2.18). Thus, using (2.55) and (2.61) we obtain

$$\|\omega^l_{,t}\|_{L^\infty(0,t*;L^2(\Omega))} \leqslant C_*, \tag{2.64}$$

where C_* is as before. It remains to deal with (2.21). In a similar way, we obtain

$$\begin{aligned}\|b^l_{,t}\|_2^2 &\leqslant C(\|\nabla b^l v^l\|_2^2 + \|\nabla(\mu^l\nabla b^l)\|_2^2 + \|\psi_t(b^l)\phi_t(\omega^l)\|_2^2 + \|\mu^l|D(v^l)|^2\|_2^2)\\ &\leqslant C(\|\nabla b^l\|_2^2\|v^l\|_\infty^2 + \|\nabla\mu^l\|_4^2\|\nabla b^l\|_4^2 + \|\mu^l\|_\infty^2\|\nabla^2 b^l\|_2^2 + \|b^l\|_\infty^2\|\omega^l\|_2^2 + \|\mu^l\|_\infty^2\|\nabla v^l\|_4^4).\end{aligned}$$

Applying again (2.55) and (2.61) we obtain

$$\|b^l_{,t}\|_{L^\infty(0,t^*;L^2(\Omega))} \leqslant C_*, \tag{2.65}$$

where C_* depends on $\Omega, c_0, t^*, \kappa_2$, $b_{\min}$, $\omega_{\max}$ and C^*.

Now, we prove the higher-order estimates for the time derivative of the approximate solution. Firstly, we multiply the equality (2.19) by $\lambda_i \frac{d}{dt} c^l_i$ and sum over i

$$(v^l_{,t}, -\Delta v^l_{,t}) + (v^l \otimes v^l, \nabla \Delta v^l_{,t}) - (\mu^l D(v^l), D(\Delta v^l_{,t})) = 0.$$

After integration by parts we get

$$\|\nabla v^l_{,t}\|_2^2 = -\left(\Delta\left(v^l \otimes v^l\right), \nabla v^l_{,t}\right) + \left(\Delta\left(\mu^l D(v^l)\right), D(v^l_{,t})\right).$$

If we apply the Hölder and Young inequalities, then we get

$$\|\nabla v^l_{,t}\|_2^2 \leqslant C(\left\|\Delta\left(v^l \otimes v^l\right)\right\|_2^2 + \left\|\Delta\left(\mu^l D(v^l)\right)\right\|_2^2),$$

where we used the equality $2\|D(v^l_{,t})\|_2^2 = \|\nabla v^l_{,t}\|_2^2$. We estimate further

$$\begin{aligned}\|\nabla v^l_{,t}\|_2^2 \leqslant C(&\left\|v^l\right\|_\infty^2 \left\|\nabla^2 v^l\right\|_2^2 + \|\nabla v^l\|_4^4 + \left\|\mu^l\right\|_\infty^2 \left\|\Delta D(v^l)\right\|_2^2) \\ &+ \|\nabla \mu^l\|_3^2 \|\nabla D(v^l)\|_6^2 + \left\|\Delta \mu^l\right\|_2^2 \left\|D(v^l)\right\|_\infty^2).\end{aligned}$$

Using the Sobolev embedding we obtain

$$\|\nabla v^l_{,t}\|_2^2 \leqslant C\Big(\|v^l\|_{2,2}^4 + \|\mu^l\|_{2,2}^2 \|v^l\|_{3,2}^2\Big),$$

where C depends only on Ω. Applying (2.55), (2.61) and (2.62) we get

$$\|\nabla v^l_{,t}\|_{L^2(0,t^*;L^2(\Omega)} \leqslant C_*, \tag{2.66}$$

where C_* depends on $c_0, \Omega, t^*, \kappa_2$, $b_{\min}$, $\omega_{\max}$ and C^*. Proceeding analogously we get

$$\|\nabla \omega^l_{,t}\|_{L^2(0,t^*;L^2(\Omega))} \leqslant C_*. \tag{2.67}$$

It remains to estimate $\nabla b^l_{,t}$. Multiplying the equality (2.21) by $\tilde{\lambda}_i \frac{d}{dt} d^l_i$ and summing over i, we obtain

$$(b^l_{,t}, -\Delta b^l_{,t}) + (b^l v^l, \nabla \Delta b^l_{,t}) - \left(\mu^l \nabla b^l, \nabla \Delta b^l_{,t}\right) = (\psi_t(b^l)\phi_t(\omega^l), \Delta b^l_{,t}) - (\mu^l |D(v^l)|^2, \Delta b^l_{,t}).$$

Integrating by parts and using the Hölder inequality yields

$$\begin{aligned} \|\nabla b^l_{,t}\|_2^2 \leqslant C(& \left\|\Delta \left(b^l v^l\right)\right\|_2 \left\|\nabla b^l_{,t}\right\|_2 + \left\|\Delta \left(\mu^l \nabla b^l\right)\right\|_2 \left\|\nabla b^l_{,t}\right\|_2 \\ &+ \left\|\nabla \left(\psi_t(b^l)\phi_t(\omega^l)\right)\right\|_2 \left\|\nabla b^l_{,t}\right\|_2 + \left\|\nabla \left(\mu^l |D(v^l)|^2\right)\right\|_2 \left\|\nabla b^l_{,t}\right\|_2). \end{aligned}$$

After applying the Young inequality we get

$$\|\nabla b^l_{,t}\|_2^2 \leqslant C(\left\|\Delta \left(b^l v^l\right)\right\|_2^2 + \left\|\Delta \left(\mu^l \nabla b^l\right)\right\|_2^2 + \left\|\nabla \left(\psi_t(b^l)\phi_t(\omega^l)\right)\right\|_2^2 + \left\|\nabla \left(\mu^l |D(v^l)|^2\right)\right\|_2^2).$$

Using the Hölder inequality we obtain

$$\begin{aligned} \|\nabla b^l_{,t}\|_2^2 \leqslant C(& \left\|\Delta b^l\right\|_2^2 \left\|v^l\right\|_\infty^2 + \|\nabla b^l\|_4^2 \|\nabla v^l\|_4^2 + \left\|b^l\right\|_\infty^2 \left\|\nabla^2 v^l\right\|_2^2 \\ &+ \left\|\Delta \mu^l\right\|_2^2 \left\|\nabla b^l\right\|_\infty^2 + \|\nabla \mu^l\|_4^2 \|\nabla^2 b^l\|_4^2 + \left\|\mu^l\right\|_\infty^2 \left\|\nabla \Delta b^l\right\|_2^2 \\ &+ \left\|\nabla(\psi_t(b^l))\right\|_2^2 \left\|\phi_t(\omega^l)\right\|_\infty^2 + \left\|\psi_t(b^l)\right\|_\infty^2 \left\|\nabla(\phi_t(\omega^l))\right\|_2^2 \\ &+ \|\nabla \mu^l\|_6^2 \|D(v^l)\|_6^4 + \left\|\mu^l\right\|_\infty^2 \|D(v^l)\|_3^2 \|\nabla D(v^l)\|_6^2). \end{aligned} \tag{2.68}$$

Applying (2.17) and (2.18) gives $\left\|\psi_t(b^l)\right\|_\infty \leqslant \left\|b^l\right\|_\infty$, $\left\|\psi_t(\omega^l)\right\|_\infty \leqslant \left\|\omega^l\right\|_\infty$ and

$$\begin{aligned} \left\|\nabla(\phi_t(\omega^l))\right\|_2 &= \left\|\phi_t'(\omega^l)\nabla \omega^l\right\|_2 \leqslant c_0 \left\|\nabla \omega^l\right\|_2, \\ \left\|\nabla(\psi_t(\omega^l))\right\|_2 &= \left\|\psi_t'(b^l)\nabla \omega^l\right\|_2 \leqslant c_0 \left\|\nabla b^l\right\|_2. \end{aligned}$$

Using these inequalities in (2.68) we obtain

$$\begin{aligned} \|\nabla b^l_{,t}\|_2^2 \leqslant C\Big(& \|b^l\|_{2,2}^2 \|v^l\|_{2,2}^2 + \|\mu^l\|_{2,2}^2 \|b^l\|_{3,2}^2 + \|\nabla b^l\|_2^2 \|\omega^l\|_{2,2}^2 + \|\nabla \omega^l\|_2^2 \|b^l\|_{2,2}^2 \\ &+ \|\mu^l\|_{2,2}^2 \|v^l\|_{2,2}^4 + \|\mu^l\|_{2,2}^2 \|v^l\|_{2,2}^2 \|v^l\|_{3,2}^2\Big), \end{aligned}$$

where $C = C(\Omega, c_0)$. Finally, from (2.55), (2.61) and (2.62) it follows that

$$\|\nabla b^l_{,t}\|_{L^2(0,t*;L^2(\Omega))} \leqslant C_*, \tag{2.69}$$

where C_* depends on $c_0, \Omega, t^*, \kappa_2$, $b_{\min}$, $\omega_{\max}$ and C^*. The estimates (2.61)-(2.65), (2.66), (2.67) and (2.69) give (2.27) and the proof of lemma 2.2.3 is finished.

□

Now, we sketch the idea of the remaining part of the proof of theorem 2.1.1. From the l-independent estimate (2.27) we deduce the existence of a subsequence, which converges weakly in some spaces (see (2.70)-(2.72)). Next, by applying the Aubin-Lions lemma we get the strong convergence of the approximate solution (see (2.73), (2.74)). Further, we prove the convergence of the "diffusive coefficient" μ^l (2.76), which allows us to take the limit in the approximate problem. As a result, we obtain (2.77)-(2.79). In the last step, we prove a series of inequalities (2.80)-(2.82), (2.84), (2.86), which show that the truncated problem is, in fact, the original one.

Having the estimate (2.27) from Lemma 2.2.3 we may apply the weak-compactness argument to the sequence of approximate solutions and we obtain a subsequence (still enumerated by superscript l) weakly convergent in appropriate spaces. To be more precise, there exist v, ω and b such that

$$v \in L^2(0,t^*;\dot{\mathcal{V}}^3_{\text{div}}) \cap L^\infty(0,t^*;\dot{\mathcal{V}}^2_{\text{div}}), \;\; v_{,t} \in L^2(0,t^*;H^1(\Omega)),$$

$$\omega, b \in L^2(0,t^*;\mathcal{V}^3) \cap L^\infty(0,t^*;\mathcal{V}^2), \;\; \omega_{,t}, b_{,t} \in L^2(0,t^*;H^1(\Omega))$$

and

$$v^l \rightharpoonup v \text{ in } L^2(0,t^*;\dot{\mathcal{V}}^3_{\text{div}}), \;\; v^l \overset{*}{\rightharpoonup} v \text{ in } L^\infty(0,t^*;\dot{\mathcal{V}}^2_{\text{div}}), \;\; v^l_{,t} \rightharpoonup v_{,t} \text{ in } L^2(0,t^*;H^1(\Omega)), \tag{2.70}$$

$$(\omega^l, b^l) \rightharpoonup (\omega, b) \text{ in } L^2(0,t^*;\mathcal{V}^3), \;\; (\omega^l, b^l) \overset{*}{\rightharpoonup} (\omega, b) \text{ in } L^\infty(0,t^*;\mathcal{V}^2), \tag{2.71}$$

$$(\omega^l_{,t}, b^l_{,t}) \rightharpoonup (\omega_{,t}, b_{,t}) \text{ in } L^2(0,t^*;H^1(\Omega)). \tag{2.72}$$

Thus, by the Aubin-Lions lemma, there exists a subsequence (again denoted by l) such that

$$(v^l, \omega^l, b^l) \longrightarrow (v, \omega, b) \quad \text{in} \quad L^2(0,t^*;H^s(\Omega)) \quad \text{for} \quad s < 3, \tag{2.73}$$

and

$$(v^l, \omega^l, b^l) \longrightarrow (v, \omega, b) \quad \text{in} \quad C([0,t^*];H^q(\Omega)) \quad \text{for} \quad q < 2. \tag{2.74}$$

We note that from (2.74) for some $\lambda > 0$ it follows

$$(v^l, \omega^l, b^l) \longrightarrow (v, \omega, b) \quad \text{in} \quad C([0, t^*]; C^{0,\lambda}(\overline{\Omega})). \tag{2.75}$$

Now, we characterise the limits of nonlinear terms. Firstly, we note that for a fixed (x, t) we may write

$$\begin{aligned}\Psi_t(b^l(x,t)) - \Psi_t(b(x,t)) &= \int_0^1 \frac{d}{ds} \left[\Psi_t\left(sb^l(x,t) + (1-s)b(x,t)\right)\right] ds \\ &= \int_0^1 \Psi_t'(sb^l(x,t) + (1-s)b(x,t))ds \cdot [b^l(x,t) - b(x,t)].\end{aligned}$$

Taking into account (2.12) we get

$$|\Psi_t(b^l(x,t)) - \Psi_t(b(x,t))| \leqslant c_0|b^l(x,t) - b(x,t)|.$$

Similarly, we obtain

$$|\Phi_t(\omega^l(x,t)) - \Phi_t(\omega(x,t))| \leqslant c_0|\omega^l(x,t) - \omega(x,t)|.$$

and

$$|\Psi_t(b(x,t))| \leqslant c_0(|b(x,t)| + b^t_{\min}).$$

Therefore, applying (2.13) we obtain

$$\begin{aligned}\left|\frac{\Psi_t(b^l)}{\Phi_t(\omega^l)} - \frac{\Psi_t(b)}{\Phi_t(\omega)}\right| &\leqslant 4(\omega^t_{\min})^{-2}\left[|\Phi_t(\omega)||\Psi_t(b^l) - \Psi_t(b)| + |\Psi_t(b)||\Phi_t(\omega) - \Phi_t(\omega^l)|\right] \\ &\leqslant C(c_0)(\omega^t_{\min})^{-2}\left[2\omega_{\max}|b^l - b| + (|b| + b^t_{\min})|\omega - \omega^l|\right].\end{aligned}$$

From (2.75) and the above estimate we have

$$\mu^l \longrightarrow \mu_{\Psi_t\Phi_t} \equiv \frac{\Psi_t(b)}{\Phi_t(\omega)} \quad \text{uniformly on} \quad \overline{\Omega} \times [0, t^*]. \tag{2.76}$$

Now, we shall take the limit $l \to \infty$ in the system (2.19)-(2.21). First, we multiply (2.19) by a_i and sum over $i \in \{1, \ldots, l\}$ and after integrating with respect to time variable we get

$$\int_0^t (v^l_{,t}, w)dt - \int_0^t (v^l \otimes v^l, \nabla w)dt + \int_0^t \left(\mu^l D(v^l), D(w)\right) dt = 0,$$

where $w = \sum_{i=1}^{l} a_i w_i$ and $t \in (0, t^*)$. Hence (2.72), (2.74) and (2.76) imply that

$$\int_0^t (v_{,t}, w)dt - \int_0^t (v \otimes v, \nabla w)dt + \int_0^t (\mu_{\Psi_t \Phi_t} D(v), D(w))\, dt = 0$$

for $t \in (0, t^*)$ and $w = \sum_{i=1}^{l} a_i w_i$. By the density argument, the above identity holds for $w \in \dot{\mathcal{V}}^1_{\mathrm{div}}$. As a consequence, we obtain

$$\int_{t_1}^{t_2} (v_{,t}, w)dt - \int_{t_1}^{t_2} (v \otimes v, \nabla w)dt + \int_{t_1}^{t_2} (\mu_{\Psi_t \Phi_t} D(v), D(w))\, dt = 0$$

for $0 < t_1 < t_2 < t^*$ and $w \in \dot{\mathcal{V}}^1_{\mathrm{div}}$. After dividing the both sides by $|t_2 - t_1|$ and taking the limit $t_2 \to t_1$ we get

$$(v_{,t}, w) - (v \otimes v, \nabla w) + (\mu_{\Psi_t \Phi_t} D(v), D(w)) = 0 \quad \text{for} \quad w \in \dot{\mathcal{V}}^1_{\mathrm{div}} \tag{2.77}$$

for a.a. $t \in (0, t^*)$. Further, we have

$$\psi_t(b^l) \longrightarrow \psi_t(b), \quad \phi_t(\omega^l) \longrightarrow \phi_t(\omega) \quad \text{uniformly on} \quad \overline{\Omega} \times [0, t^*].$$

Thus, using (2.20) and (2.21) and arguing as earlier we obtain

$$(\omega_{,t}, z) - (\omega v, \nabla z) + (\mu_{\Psi_t \Phi_t} \nabla \omega, \nabla z) = -\kappa_2(\phi_t^2(\omega), z) \quad \text{for} \quad z \in \mathcal{V}^1, \tag{2.78}$$

$$(b_{,t}, q) - (bv, \nabla q) + (\mu_{\Psi_t \Phi_t} \nabla b, \nabla q) = -(\psi_t(b)\phi_t(\omega), q) + (\mu_{\Psi_t \Phi_t} |D(v)|^2, q) \text{ for } q \in \mathcal{V}^1 \tag{2.79}$$

for a.a. $t \in (0, t^*)$.

Now, we shall prove the bounds for b and ω. The proof is similar to one found in [36]. We denote by b_+ (b_-) the positive (negative resp.) part of b. Then $b = b_+ + b_-$. We shall show that

$$b \geqslant 0 \quad \text{in} \quad \overline{\Omega} \times [0, t^*]. \tag{2.80}$$

For this purpose we test the equation (2.79) by b_- and obtain

$$(b_{,t}, b_-) - (bv, \nabla b_-) + (\mu_{\Psi_t \Phi_t} \nabla b, \nabla b_-) = -(\psi_t(b)\phi_t(\omega), b_-) + (\mu_{\Psi_t \Phi_t} |D(v)|^2, b_-).$$

We note that from (2.76) we have $0 \leqslant \mu_{\Psi_t\Phi_t}$ and by (2.14) we obtain $\psi_t(b)b_- \equiv 0$. Thus we get

$$(\partial_t b_-, b_-) - (b_- v, \nabla b_-) + (\mu_{\Psi_t\Phi_t}\nabla b_-, \nabla b_-) \leqslant 0$$

and then

$$\frac{d}{dt}\|b_-\|_2^2 \leqslant 0.$$

By the assumption (2.1) the negative part of the initial value of b is zero hence, $b_- \equiv 0$ and we obtained (2.80).

Proceeding similarly we introduce the decomposition $\omega = \omega_+ + \omega_-$ and test the equation (2.78) by ω_-

$$(\omega_{,t}, \omega_-) - (\omega v, \nabla\omega_-) + (\mu_{\Psi_t\Phi_t}\nabla\omega, \nabla\omega_-) = -(\phi_t^2(\omega), \omega_-).$$

We note that by (2.15) the right-hand side of the above equality vanishes. Thus we get $\frac{d}{dt}\|\omega_-\|_2^2 \leqslant 0$ and by assumption (2.2)

$$\omega \geqslant 0 \quad \text{in} \quad \overline{\Omega} \times [0, t^*]. \tag{2.81}$$

Now, we shall prove that

$$\omega(x,t) \geqslant \frac{\omega_{\min}}{1 + \kappa_2\omega_{\min}t} \quad \text{for} \quad (x,t) \in \overline{\Omega} \times [0, t^*]. \tag{2.82}$$

We test the equation (2.78) by $(\omega - \omega_{\min}^t)_-$ and obtain

$$\begin{aligned}(\omega_{,t}, (\omega - \omega_{\min}^t)_-) - (\omega v, \nabla(\omega - \omega_{\min}^t)_-) + \Big(\mu_{\Psi_t\Phi_t}\nabla\omega, \nabla\left(\omega - \omega_{\min}^t\right)_-\Big)& \\ = -\kappa_2(\phi_t^2(\omega), (\omega - \omega_{\min}^t)_-).&\end{aligned} \tag{2.83}$$

Using (2.3) we get

$$(\omega_{,t}, (\omega - \omega_{\min}^t)_-) = \frac{1}{2}\frac{d}{dt}\|(\omega - \omega_{\min}^t)_-\|_2^2 - \kappa_2\left((\omega_{\min}^t)^2, (\omega - \omega_{\min}^t)_-\right).$$

Hence using inequality $0 \leqslant \mu_{\Psi_t\Phi_t}$ and $\operatorname{div} v = 0$ in (2.83) yields

$$\frac{1}{2}\frac{d}{dt}\|(\omega - \omega_{\min}^t)_-\|_2^2 - \kappa_2\left((\omega_{\min}^t)^2, (\omega - \omega_{\min}^t)_-\right) \leqslant -\kappa_2(\phi_t^2(\omega), (\omega - \omega_{\min}^t)_-).$$

We write the above inequality in the form

$$\frac{1}{2}\frac{d}{dt}\|(\omega-\omega^t_{\min})_-\|_2^2 \leqslant -\kappa_2((\phi_t(\omega)-\omega^t_{\min})(\phi_t(\omega)+\omega^t_{\min}),(\omega-\omega^t_{\min})_-).$$

We note that $-\kappa_2((\phi_t(\omega)+\omega^t_{\min}),(\omega-\omega^t_{\min})_-)$ is non-negative. Thus using (2.18) we get $\phi_t(\omega)\leqslant\omega$, so

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(\omega-\omega^t_{\min})_-\|_2^2 &\leqslant -\kappa_2((\omega-\omega^t_{\min})(\phi_t(\omega)+\omega^t_{\min}),(\omega-\omega^t_{\min})_-)\\ &= -\kappa_2\big((\phi_t(\omega)+\omega^t_{\min}),\big|(\omega-\omega^t_{\min})_-\big|^2\big)\leqslant 0.\end{aligned}$$

Therefore, we obtain $\frac{d}{dt}\|(\omega-\omega^t_{\min})_-\|_2^2\leqslant 0$ and by (2.2) we get (2.82).
Now, we shall prove that

$$\omega(x,t)\leqslant\frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}\quad\text{for}\quad(x,t)\in\overline{\Omega}\times[0,t^*]. \tag{2.84}$$

Indeed, firstly we note that from (2.3), (2.15) and (2.82) we have

$$\phi_t(\omega)=\omega. \tag{2.85}$$

Hence, testing the equation (2.78) by $(\omega-\omega^t_{\max})_+$ gives

$$\begin{aligned}(\omega_{,t},(\omega-\omega^t_{\max})_+)-(\omega v,\nabla(\omega-\omega^t_{\max})_+)+\Big(\mu_{\Psi_t\Phi_t}\nabla\omega,\nabla\left(\omega-\omega^t_{\max}\right)_+\Big)&\\ =-\kappa_2(\omega^2,(\omega-\omega^t_{\max})_+).&\end{aligned}$$

Proceeding as before, we get

$$\frac{1}{2}\frac{d}{dt}\|(\omega-\omega^t_{\max})_+\|_2^2-\kappa_2\left((\omega^t_{\max})^2,(\omega-\omega^t_{\max})_+\right)\leqslant-\kappa_2(\omega^2,(\omega-\omega^t_{\max})_+).$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(\omega-\omega^t_{\max})_+\|_2^2 &\leqslant -\kappa_2((\omega-\omega^t_{\max})(\omega+\omega^t_{\max}),(\omega-\omega^t_{\max})_+)\\ &= -\kappa_2((\omega+\omega^t_{\max}),|(\omega-\omega^t_{\max})_+|^2).\end{aligned}$$

Hence

$$\frac{1}{2}\frac{d}{dt}\|(\omega-\omega_{\max}^t)_+\|_2^2 \leqslant 0.$$

By (2.2) we get (2.84). We shall prove that

$$b(x,t) \geqslant b_{\min}^t \quad \text{for} \quad (x,t) \in \overline{\Omega} \times [0,t^*]. \tag{2.86}$$

For this purpose we test the equation (2.79) by $(b-b_{\min}^t)_-$. Then we get

$$\begin{aligned}(b_{,t},(b-b_{\min}^t)_-) - (bv, \nabla((b-b_{\min}^t)_-)) + \big(\mu_{\Psi_t\Phi_t}\nabla b, \nabla((b-b_{\min}^t)_-)\big) \\ = -(\psi_t(b)\omega,(b-b_{\min}^t)_-) + (\mu_{\Psi_t\Phi_t}\left|D(v)\right|^2,(b-b_{\min}^t)_-).\end{aligned}$$

The first term on the left-hand side is equal to

$$\frac{1}{2}\frac{d}{dt}\|(b-b_{\min}^t)_-\|_2^2 - \left(\frac{\omega_{\max}b_{\min}}{(1+\omega_{\max}\kappa_2 t)^{\frac{1}{\kappa_2}+1}},(b-b_{\min}^t)_-\right).$$

The second term of the left-hand side vanishes and the third one is non-negative. Thus, it follows that

$$\frac{1}{2}\frac{d}{dt}\|(b-b_{\min}^t)_-\|_2^2 - \left(\frac{\omega_{\max}b_{\min}}{(1+\omega_{\max}\kappa_2 t)^{\frac{1}{\kappa_2}+1}},(b-b_{\min}^t)_-\right) \leqslant -(\psi_t(b)\omega,(b-b_{\min}^t)_-).$$

Using (2.84) we obtain

$$\frac{1}{2}\frac{d}{dt}\|(b-b_{\min}^t)_-\|_2^2 - \left(\frac{\omega_{\max}b_{\min}}{(1+\omega_{\max}\kappa_2 t)^{\frac{1}{\kappa_2}+1}},(b-b_{\min}^t)_-\right) \leqslant -\frac{\omega_{\max}}{1+\omega_{\max}\kappa_2 t}(\psi_t(b),(b-b_{\min}^t)_-)$$

and by definition (2.3) we get

$$\frac{1}{2}\frac{d}{dt}\|(b-b_{\min}^t)_-\|_2^2 \leqslant -\frac{\omega_{\max}}{1+\omega_{\max}\kappa_2 t}(\psi_t(b)-b_{\min}^t,(b-b_{\min}^t)_-).$$

From (2.80) and (2.17) we have that $\psi_t(b) \leqslant b$, so

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(b-b_{\min}^t)_-\|_2^2 &\leqslant -\frac{\omega_{\max}}{1+\omega_{\max}\kappa_2 t}(b-b_{\min}^t,(b-b_{\min}^t)_-) \\ &= -\frac{\omega_{\max}}{1+\omega_{\max}\kappa_2 t}\|(b-b_{\min}^t)_-\|_2^2\end{aligned}$$

and then $\frac{d}{dt}\|(b - b^t_{\min})_-\|_2^2 \leqslant 0$. Using (2.1) and (2.3) we get (2.86).

Note that (2.14) and (2.86) imply

$$\psi_t(b) = b. \tag{2.87}$$

Furthermore, (2.11) and (2.86) give $\Psi_t(b) = b$. Finally, (2.3), (2.13), (2.82) and (2.84) yield $\Phi_t(\omega) = \omega$. Thus,

$$\mu_{\Psi_t\Phi_t} = \frac{\Psi_t(b)}{\Phi_t(\omega)} = \frac{b}{\omega}. \tag{2.88}$$

Applying (2.85), (2.87) and (2.88) we deduce that system (2.77)-(2.79) has the following form

$$(v_{,t}, w) - (v \otimes v, \nabla w) + \left(\frac{b}{\omega} D(v), D(w)\right) = 0 \quad \text{for} \quad w \in \dot{\mathcal{V}}^1_{\text{div}}, \tag{2.89}$$

$$(\omega_{,t}, z) - (\omega v, \nabla z) + \left(\frac{b}{\omega} \nabla\omega, \nabla z\right) = -\kappa_2(\omega^2, z) \quad \text{for} \quad z \in \mathcal{V}^1, \tag{2.90}$$

$$(b_{,t}, q) - (bv, \nabla q) + \left(\frac{b}{\omega} \nabla b, \nabla q\right) = -(b\omega, q) + \left(\frac{b}{\omega} |D(v)|^2, q\right) \quad \text{for} \quad q \in \mathcal{V}^1 \tag{2.91}$$

for a.a. $t \in (0, t^*)$.

Chapter 3

Global in time solution for small initial data

In this chapter the existence of a global-in-time, regular solution will be proven under a certain smallness condition. The basic idea behind the formulated smallness condition is to guarantee the small oscillations of initial data in comparison to the turbulent viscosity $\frac{b}{\omega}$. The detailed formulation of the result is given in Theorem 3.2.1. Corollary 3.2.4.1 shows that the formulated condition is fulfilled by a non-nonempty class of functions. Results presented in this chapter are published in [30].

3.1. Notation and notion of a solution

Assume that $\Omega = \prod_{i=1}^{3}(0, L_i)$, L_i, $T > 0$ and $\Omega^T = \Omega \times (0, T)$. We shall consider the problem (1)-(5) in Ω^T. Constants $\nu_0, \kappa_1, \ldots, \kappa_4$ are positive. For simplicity, we assume further that all constants except κ_2 are equal to one. The reason is that the constant κ_2 plays an important role in a priori estimates.

We shall show the global-in-time existence of a regular solution of problem (1)-(5) under some assumption imposed on the initial data. Firstly, suppose that $v_0 \in \dot{\mathcal{V}}^2_{\mathrm{div}}$, ω_0, $b_0 \in \mathcal{V}^2$ and that there exist positive numbers $b_{\min}$, $\omega_{\min}$, $\omega_{\max}$ such that

$$0 < b_{\min} \leqslant b_0(x), \tag{3.1}$$

$$0 < \omega_{\min} \leqslant \omega_0(x) \leqslant \omega_{\max} \tag{3.2}$$

on Ω. Now, we will introduce notation that will enable us to formulate the smallness condition from Theorem 3.2.1. Based on the introduced bound we define

$$\omega_{\min}^t = \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t}, \quad \omega_{\max}^t = \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}. \tag{3.3}$$

These quantities will appear in the lower and upper bounds for ω (see Proposition 3.3.1). Additionally, we introduce the analogous notation for b, for the lower bound of b and the upper bound of $\|b\|_1$ (see Proposition 3.3.1 and Proposition 3.3.3c)

$$b_{\min}^t = \frac{b_{\min}}{(1+\kappa_2\omega_{\max}t)^{\frac{1}{\kappa_2}}}, \; b_{\max}(t) = \frac{\|b_0\|_1 + \frac{1}{2}\|v_0\|_2^2\left(1+I_\infty\left(\kappa_2, \frac{\omega_{\min}}{\omega_{\max}}, \frac{b_{\min}}{(\omega_{\max})^2}\right)\right)}{(1+\kappa_2\omega_{\min}t)^{\frac{1}{\kappa_2}}}, \tag{3.4}$$

where

$$I_\infty(\kappa_2, x, y) = \Gamma\left(\frac{2\kappa_2}{2\kappa_2-1}\right) x^{\min\left\{1,\frac{1}{\kappa_2}\right\}} \left(\frac{C_p^2(2\kappa_2-1)}{y}\exp\left(\frac{y}{C_p^2}\right)\right)^{\frac{1}{2\kappa_2-1}}, \tag{3.5}$$

and C_p is the Poincaré constant for the domain Ω, i.e. the smallest constant such that $\|f\|_p \leqslant C_p\|\nabla f\|_p$ for smooth f such that $\int_\Omega f dx = 0$. In the case of b we will be able to control the decay of L^1-norm. Frequently we will estimate from below the coefficient in the diffusive term by (see (3.63))

$$\mu_{\min}^t = \frac{b_{\min}^t}{\omega_{\max}^t} = \frac{b_{\min}}{\omega_{\max}}(1+\kappa_2\omega_{\max}t)^{1-\frac{1}{\kappa_2}}. \tag{3.6}$$

To express the smallness of the initial data we will need the following quantity

$$\begin{aligned} Y_2(t) = \left(\|\Delta b_0\|_2^2 + \|\Delta\omega_0\|_2^2 + \|\Delta v_0\|_2^2\right)\cdot \\ \cdot\exp\left(-\frac{1}{C_p^2}\frac{b_{\min}}{(2\kappa_2-1)\omega_{\max}^2}\left((1+\kappa_2\omega_{\max}t)^{2-1/\kappa_2}-1\right)\right). \end{aligned} \tag{3.7}$$

Furthermore, to formulate a condition that ensures the existence of a global-in-time solution we have to define (see (3.16) in Theorem 3.2.1)

$$A(t) = \left(\|v_0\|_2^2\exp\left(-\frac{b_{\min}\left((1+\kappa_2\omega_{\max}t)^{2-\frac{1}{\kappa_2}}-1\right)}{C_p^2\omega_{\max}^2(2\kappa_2-1)}\right) + b_{\max}(t)^2\right)^{\frac{1}{4}}, \tag{3.8}$$

$$B(t) = 1 + \frac{1}{\omega^t_{\min}} + \frac{b_{\max}(t)}{\omega^t_{\min}} + \frac{b_{\max}(t)}{(\omega^t_{\min})^2}, \tag{3.9}$$

$$C(t) = \frac{1}{\omega^t_{\min}} + \frac{1}{(\omega^t_{\min})^2} + \frac{b_{\max}(t)}{(\omega^t_{\min})^2} + \frac{b_{\max}(t)}{(\omega^t_{\min})^3}, \tag{3.10}$$

$$D(t) = \frac{1}{(\omega^t_{\min})^2} + \frac{1}{(\omega^t_{\min})^3} \tag{3.11}$$

and

$$Z_0(t) = \Big(b_{\max}(t) + A(t)Y_2^{\frac{1}{4}}(t) + B(t)Y_2^{\frac{1}{2}}(t) + C(t)Y_2(t) + D(t)Y_2^{\frac{3}{2}}(t)\Big). \tag{3.12}$$

Now, we introduce the notion of a solution to the system (1)-(5). For $v_0 \in \dot{\mathcal{V}}^2_{\mathrm{div}}$ and strictly positive ω_0, $b_0 \in \mathcal{V}^2$, functions $(v, \omega, b) \in \mathcal{X}(\infty)$ are global solution to (1)-(5) if

$$(v_{,t}, w) - (v \otimes v, \nabla w) + (\mu D(v), D(w)) = 0 \quad \text{for} \quad w \in \dot{\mathcal{V}}^1_{\mathrm{div}}, \tag{3.13}$$

$$(\omega_{,t}, z) - (\omega v, \nabla z) + (\mu \nabla \omega, \nabla z) = -\kappa_2(\omega^2, z) \quad \text{for} \quad z \in \mathcal{V}^1, \tag{3.14}$$

$$(b_{,t}, q) - (bv, \nabla q) + (\mu \nabla b, \nabla q) = -(b\omega, q) + (\mu |D(v)|^2, q) \quad \text{for} \quad q \in \mathcal{V}^1 \tag{3.15}$$

for a.a. $t \in (0, \infty)$, where $\mu = \frac{b}{\omega}$ and (5) holds. Recall that $D(v)$ denotes the symmetric part of ∇v and $(\cdot, \cdot)$ is the inner product in $L^2(\Omega)$.

3.2. Main result

Now, we formulate the main result involving the global existence of a regular solution to system (1)-(5).

Theorem 3.2.1. *Assume that $\kappa_2 > \frac{1}{2}$. There exists a constant C_{Ω,κ_2}, which depends only on Ω and κ_2, with the following property: for any ω_0, $b_0 \in \mathcal{V}^2$, $v_0 \in \dot{\mathcal{V}}^2_{\mathrm{div}}$, if (3.1), (3.2) hold and*

$$\mu^t_{\min} - C_{\Omega,\kappa_2} Z_0(t) > 0 \qquad \text{for} \quad t \in [0, T), \tag{3.16}$$

for some $T \in (0, \infty]$, then there exists a unique solution $(v, \omega, b) \in \mathcal{X}(T)$ to (1)-(5) in Ω^T.

We recall that we assume that the constants ν_0, κ_1, κ_3 and κ_4 are equal to one. In the general case, if all these constants are positive and arbitrary, then the constant in the above result will depend on ν_0, κ_1, ..., κ_4 and Ω. The functions $\mu^t_{\min}$ and $Z_0(t)$ were defined in (3.6) and (3.12), respectively.

Remark 3.2.2. *The condition (3.16) involves only the initial data: v_0, ω_0, b_0, the parameters of the system: ν_0, κ_1, ..., κ_4 and Ω.*

Remark 3.2.3. *The assumption $\kappa_2 > \frac{1}{2}$ is crucial in the proof of Theorem 3.2.1 (and also in Proposition 3.3.3). Without it, we are unable to prove the exponential decay of L^2-norm of $v(t)$ and polynomial decay of L^1-norm of $b(t)$. These decay rates play an important role in the presented proof.*

Remark 3.2.4. *As is stated in [45], Kolmogorov set $\kappa_2 = \frac{7}{11}$ and Theorem 3.2.1 may be applied for this value of parameter κ_2.*

As a consequence of theorem 3.2.1 we have

Corollary 3.2.4.1. *Assume that $\kappa_2 > \frac{1}{2}$, $v_0 \in \dot{\mathcal{V}}^2_{\text{div}}$, ω_0, $b_0 \in \mathcal{V}^2$ and the conditions (3.1), (3.2) hold. We denote*

$$a_0 = \sup_{t \geqslant 0} 2C_{\Omega,\kappa_2}(1+\kappa_2\omega_{\max}t)^{\frac{1}{\kappa_2}-1}\left(A(t) + B(t)Y_2^{\frac{1}{4}}(t) + C(t)Y_2^{\frac{3}{4}}(t) + D(t)Y_2^{\frac{5}{4}}(t)\right),$$

where C_{Ω,κ_2} is the constant given in theorem 3.2.1 and $Y_2, A(t), \ldots, D(t)$ were defined in (3.7)-(3.11). Then a_0 is finite. If in addition,

$$\frac{b_{\min}}{\omega_{\max}} > 2C_{\Omega,\kappa_2}\left(\|b_0\|_1 + \frac{1}{2}\|v_0\|_2^2\left(1 + I_\infty\left(\kappa_2, \frac{\omega_{\min}}{\omega_{\max}}, \frac{b_{\min}}{(\omega_{\max})^2}\right)\right)\right) \quad \text{for } \kappa_2 \geqslant 1 \tag{3.17}$$

and for $\kappa_2 \in \left(\frac{1}{2}, 1\right)$

$$\frac{b_{\min}}{\omega_{\max}} > 2C_{\Omega,\kappa_2}\left(\|b_0\|_1 + \frac{1}{2}\|v_0\|_2^2\left(1 + I_\infty\left(\kappa_2, \frac{\omega_{\min}}{\omega_{\max}}, \frac{b_{\min}}{(\omega_{\max})^2}\right)\right)\right)\left(\frac{\omega_{\max}}{\omega_{\min}}\right)^{\frac{1}{\kappa_2}} \tag{3.18}$$

and

$$\frac{b_{\min}}{\omega_{\max}} > a_0\left(\|\Delta v_0\|_2^2 + \|\Delta\omega_0\|_2^2 + \|\Delta b_0\|_2^2\right)^{\frac{1}{4}} \tag{3.19}$$

hold, then the system (1)-(5) has a unique global solution in $\mathcal{X}(\infty)$.

Remark 3.2.5. *The conditions (3.17)-(3.19) involve only the initial data: v_0, ω_0, b_0, the parameters of the system: ν_0, κ_1, ..., κ_4 and Ω.*

Remark 3.2.6. *We shall show that the conditions (3.17)-(3.19) are satisfied on some non-empty set of initial data. We focus only on the case* $\kappa_2 \in (\frac{1}{2}, 1)$*, because the other is simpler. It may be done in the following way: we shall determine positive* δ_1,δ_2,δ_3 *such that if initial data satisfy the bounds*

$$\|b_0\|_1 \leqslant \delta_1, \ \ \|v_0\|_2 \leqslant \delta_2, \ \ \|\Delta v_0\|_2^2 + \|\Delta \omega_0\|_2^2 + \|\Delta b_0\|_2^2 \leqslant \delta_3, \tag{3.20}$$

then (3.18) and (3.19) will be fulfilled. We proceed in the following way:

— *set* $\omega_{\min}$ *and* $\omega_{\max}$ *such that* $0 < \omega_{\min} < \omega_{\max}$ *and*

$$2C_{\Omega,\kappa_2}|\Omega|(\omega_{\max})^{1+\frac{1}{\kappa_2}} < (\omega_{\min})^{\frac{1}{\kappa_2}},$$

i.e.

$$\frac{1}{\omega_{\max}} > 2C_{\Omega,\kappa_2}|\Omega|\left(\frac{\omega_{\max}}{\omega_{\min}}\right)^{\frac{1}{\kappa_2}},$$

— *fix* $b_{\min} > 0$ *so, we have*

$$\frac{b_{\min}}{\omega_{\max}} > 2C_{\Omega,\kappa_2}b_{\min}|\Omega|\left(\frac{\omega_{\max}}{\omega_{\min}}\right)^{\frac{1}{\kappa_2}},$$

— *choose* $\delta_1 > b_{\min}|\Omega|$ *such that*

$$\frac{b_{\min}}{\omega_{\max}} > 2C_{\Omega,\kappa_2}\delta_1\left(\frac{\omega_{\max}}{\omega_{\min}}\right)^{\frac{1}{\kappa_2}},$$

— *find* $\delta_2 > 0$ *such that*

$$\frac{b_{\min}}{\omega_{\max}} > 2C_{\Omega,\kappa_2}\left(\delta_1 + \frac{1}{2}\delta_2\left(1 + I_\infty\left(\kappa_2, \frac{\omega_{\min}}{\omega_{\max}}, \frac{b_{\min}}{(\omega_{\max})^2}\right)\right)\right)\left(\frac{\omega_{\max}}{\omega_{\min}}\right)^{\frac{1}{\kappa_2}},$$

— *if we define* $a_0(\delta_1, \delta_2, \delta_3)$ *similarly as in Corollary 3.2.4.1, where we replace* $\|b_0\|_1$ *by* δ_1*,* $\|v_0\|_2$ *by* δ_2 *and* $\|\Delta v_0\|_2^2 + \|\Delta \omega_0\|_2^2 + \|\Delta b_0\|_2^2$ *by* δ_3*, then from (3.4), (3.7) and (3.8)-(3.11) we deduce that* $a_0(\delta_1, \delta_2, \delta_3)$ *is increasing with respect to each* δ_i*. Therefore, we can find* $\delta_3 > 0$ *such that*

$$\frac{b_{\min}}{\omega_{\max}} > a_0(\delta_1, \delta_2, \delta_3)\delta_3^{\frac{1}{4}},$$

— *finally, for these positive numbers* $\delta_1, \delta_2, \delta_3$ *and any* b_0, ω_0 *and* v_0 *such that* $b_{\min} \leqslant b_0$, $\omega_{\min} \leqslant \omega_0 \leqslant \omega_{\max}$ *and (3.20) hold, the conditions (3.18) and (3.19) are satisfied.*

3.3. Proof of Theorem 3.2.1

We need the following auxiliary results (see also theorem 4.1 [36]).

Proposition 3.3.1. *Assume that* ω_0, $b_0 \in \mathcal{V}^2$, $v_0 \in \dot{\mathcal{V}}^2_{\mathrm{div}}$ *and (3.1), (3.2) hold. If* $T > 0$ *and* $(v, \omega, b) \in \mathcal{X}(T)$ *satisfies (1)-(5), then the following estimates*

$$\frac{\omega_{\min}}{1 + \kappa_2 \omega_{\min} t} \leqslant \omega(x,t) \leqslant \frac{\omega_{\max}}{1 + \kappa_2 \omega_{\max} t}, \tag{3.21}$$

$$\frac{b_{\min}}{(1 + \kappa_2 \omega_{\max} t)^{\frac{1}{\kappa_2}}} \leqslant b(x,t) \tag{3.22}$$

hold for $(x,t) \in \Omega^T$.

Proof. By the assumption we have $\omega, b \in L^2_{loc}([0,T]; H^3(\Omega))$, $\omega_{,t}, b_{,t} \in L^2_{loc}([0,T]; H^1(\Omega))$. Thus the Sobolev embedding theorem implies that $\omega, b \in C(\overline{\Omega} \times [0,T))$. Then, by (3.1) and (3.2) there exists $t_1 \in (0,T)$ such that

$$\frac{1}{2} b_{\min} \leqslant b(x,t), \ \ \frac{1}{2}\omega_{\min} \leqslant \omega(x,t) \leqslant 2\omega_{\max} \quad \text{for} \quad (x,t) \in \Omega^{t_1}. \tag{3.23}$$

We denote by f_+ and f_- the non-negative and non-positive parts of function f, i.e. $f = f_+ + f_-$, where $f_+ = \max\{f, 0\}$. For $t \in (0, t_1)$ we test the equality (3.14) by $z = (\omega - \omega^t_{\min})_-$ and we obtain

$$(\omega_{,t}, (\omega - \omega^t_{\min})_-) + \left(\frac{b}{\omega}\nabla\omega, \nabla(\omega - \omega^t_{\min})_-\right) = -\kappa_2(\omega^2, (\omega - \omega^t_{\min})_-),$$

where we used the condition $\operatorname{div} v = 0$. Using the equality $(\omega^t_{\min})_{,t} = -\kappa_2(\omega^t_{\min})^2$ we may write

$$\frac{1}{2}\frac{d}{dt}\|(\omega - \omega^t_{\min})_-\|_2^2 - \kappa_2\left((\omega^t_{\min})^2, (\omega - \omega^t_{\min})_-\right) + \left(\frac{b}{\omega}\nabla(\omega - \omega^t_{\min})_-, \nabla(\omega - \omega^t_{\min})_-\right)$$
$$= -\kappa_2(\omega^2, (\omega - \omega^t_{\min})_-)$$

for $t \in (0, t_1)$. After applying (3.23) we get

$$\frac{1}{2}\frac{d}{dt}\|(\omega - \omega^t_{\min})_-\|_2^2 \leqslant -\kappa_2 \left((\omega - \omega^t_{\min})(\omega + \omega^t_{\min}), (\omega - \omega^t_{\min})_-\right)$$
$$= -\kappa_2 \left(\omega + \omega^t_{\min}, \left|(\omega - \omega^t_{\min})_-\right|^2\right).$$

By the Grönwall inequality and (3.2) we deduce that $(\omega - \omega^t_{\min})_- \equiv 0$ on for $t \in (0, t_1)$. Hence

$$\frac{\omega_{\min}}{1 + \kappa_2 \omega_{\min} t} \leqslant \omega(x, t) \tag{3.24}$$

for $(x, t) \in \overline{\Omega} \times [0, t_1)$. Next, testing the equation (3.14) by $z = (\omega - \omega^t_{\max})_+$ and proceeding as in (3.24) we deduce that

$$\omega(x, t) \leqslant \frac{\omega_{\max}}{1 + \kappa_2 \omega_{\max} t} \tag{3.25}$$

for $(x, t) \in \overline{\Omega} \times [0, t_1)$. Now, for $t \in (0, t_1)$ we test the equation (3.15) by $q = (b - b^t_{\min})_-$ and we obtain

$$(b_{,t}, (b - b^t_{\min})_-) + \left(\frac{b}{\omega}\nabla(b - b^t_{\min})_-, \nabla(b - b^t_{\min})_-\right) = -(b\omega, (b - b^t_{\min})_-)$$
$$+ \left(\frac{b}{\omega}\left|D(v)\right|^2, (b - b^t_{\min})_-\right),$$

where we used the condition $\operatorname{div} v = 0$. By applying (3.23) we get

$$(b_{,t}, (b - b^t_{\min})_-) \leqslant -(b\omega, (b - b^t_{\min})_-),$$

i.e.

$$\frac{1}{2}\frac{d}{dt}\|(b - b^t_{\min})_-\|_2^2 - \frac{\omega_{\max}}{(1 + \kappa_2 \omega_{\max} t)}\left(b^t_{\min}, (b - b^t_{\min})_-\right) \leqslant -(b\omega, (b - b^t_{\min})_-).$$

From (3.23) and (3.25) we obtain

$$-(b\omega, (b - b^t_{\min})_-) \leqslant -\frac{\omega_{\max}}{(1 + \kappa_2 \omega_{\max} t)}(b, (b - b^t_{\min})_-)$$

for $t \in (0, t_1)$. Hence

$$\frac{1}{2}\frac{d}{dt}\|(b - b^t_{\min})_-\|_2^2 \leqslant -\frac{\omega_{\max}}{(1 + \kappa_2 \omega_{\max} t)}(b - b^t_{\min}, (b - b^t_{\min})_-).$$

The right-hand side is non-positive. Thus, by (3.1) we have

$$\frac{b_{\min}}{(1+\kappa_2\omega_{\max}t)^{\frac{1}{\kappa_2}}} \leqslant b(x,t) \tag{3.26}$$

for $(x,t) \in \overline{\Omega} \times [0,t_1)$. Now, we define

$$t_1^* = \sup\{\widetilde{t} \in (0,T) : (3.21), (3.22) \quad \text{hold for } (x,t) \in \Omega^{\widetilde{t}}\}.$$

By the previous step, we have $t_1^* \geqslant t_1 > 0$. If $t_1^* < T$, then by the continuity of ω, b and (3.24)-(3.26) there exists $t_2 \in (t_1^*, T)$ such that

$$\frac{1}{2}b^t_{\min} \leqslant b(x,t), \quad \frac{1}{2}\omega^t_{\min} \leqslant \omega(x,t) \leqslant 2\omega^t_{\max} \quad \text{for } (x,t) \in \Omega^{t_2}.$$

Then, we have $\frac{b(x,t)}{\omega(x,t)} \geqslant \frac{1}{4}\frac{b^t_{\min}}{\omega^t_{\max}} > 0$ for $(x,t) \in \Omega \times [0,t_2)$ and we may repeat the argument from the first part of the proof and as a consequence we get $t_2 \leqslant t_1^*$. This contradiction means that $t_1^* = T$ and the proof is finished. □

Proposition 3.3.2. *For any $T > 0$, the problem (1)-(5) has at most one solution in $\mathcal{X}(T)$.*

Proof. Suppose that (v^1, ω^1, b^1), $(v^2, \omega^2, b^2) \in \mathcal{X}(T)$ satisfy (1)-(5) in Ω^T. We denote $v = v^1 - v^2$, $\omega = \omega^1 - \omega^2$, $b = b^1 - b^2$ and we test the equations for v^1 and v^2 by v. After subtracting the equations for v^i we get

$$(v_{,t}, v) - \left(v^1 \otimes v^1 - v^2 \otimes v^2, \nabla v\right) + \left(\frac{b^1}{\omega^1}D(v^1) - \frac{b^2}{\omega^2}D(v^2), D(v)\right) = 0.$$

We note that

$$\begin{aligned}\left(\frac{b^1}{\omega^1}D(v^1) - \frac{b^2}{\omega^2}D(v^2), D(v)\right) = &\left(\frac{b^1}{\omega^1}D(v), D(v)\right) + \left(\frac{b}{\omega^1}D(v^2), D(v)\right) \\ &- \left(\frac{b^2\omega}{\omega^1\omega^2}D(v^2), D(v)\right),\end{aligned}$$

$$\left(v^1 \otimes v^1 - v^2 \otimes v^2, \nabla v\right) = \left(v^1 \otimes v, \nabla v\right) + \left(v \otimes v^2, \nabla v\right).$$

By Proposition 3.3.1 we have $\frac{b^1}{\omega^1} \geqslant \mu^t_{\min}$, so by the Hölder inequality we get

$$\frac{1}{2}\frac{d}{dt}\|v\|_2^2 + \mu^t_{\min}\|D(v)\|_2^2 \leqslant \left\|\frac{1}{\omega^1}\right\|_\infty \|b\|_2 \left\|D(v^2)\right\|_\infty \|D(v)\|_2$$
$$+ \left\|\frac{1}{\omega^1\omega^2}\right\|_\infty \|b^2\|_\infty \|\omega\|_2 \left\|D(v^2)\right\|_\infty \|D(v)\|_2 + \|v^1\|_\infty \|v\|_2 \|\nabla v\|_2 + \|v\|_2 \|v^2\|_\infty \|\nabla v\|_2 .$$

By Proposition 3.3.1 functions ω^1 and ω^2 are estimated from below by $\omega^t_{\min}$. Therefore the Young inequality, the Sobolev embedding theorem and $\|D(v)\|_2 = \frac{\sqrt{2}}{2}\|\nabla v\|_2$ imply

$$\begin{aligned}\frac{d}{dt}\|v\|_2^2 + \mu^t_{\min}\|D(v)\|_2^2 \leqslant \frac{C}{\mu^t_{\min}}\Big(&(\omega^t_{\min})^{-2}\|v^2\|_{3,2}^2\|b\|_2^2 + (\omega^t_{\min})^{-4}\|b^2\|_{2,2}^2\|v^2\|_{3,2}^2\|\omega\|_2^2 \\ &+ \left(\|v^1\|_{2,2}^2 + \|v^2\|_{2,2}^2\right)\|v\|_2^2\Big),\end{aligned} \tag{3.27}$$

where C depends only on Ω. Now we test the equations for ω^1 and ω^2 by $\omega = \omega^1 - \omega^2$ and as a result we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\omega\|_2^2 + \left(\frac{b^1}{\omega^1}\nabla\omega, \nabla\omega\right) &= \left(\omega^1 v, \nabla\omega\right) + \left(\omega v^2, \nabla\omega\right) - \left(\frac{b}{\omega^1}\nabla\omega^2, \nabla\omega\right) \\ &+ \left(\frac{b^2\omega}{\omega^1\omega^2}\nabla\omega^2, \nabla\omega\right) - \kappa_2\left(\omega(\omega^1+\omega^2), \omega\right).\end{aligned}$$

From the Hölder inequality, (3.21) and (3.22) we get

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_2^2 + \mu^t_{\min}\|\nabla\omega\|_2^2 \leqslant \|\omega^1\|_\infty \|v\|_2 \|\nabla\omega\|_2 + \|\omega\|_2 \|v^2\|_\infty \|\nabla\omega\|_2$$
$$+ \left\|\frac{1}{\omega^1}\right\|_\infty \|b\|_2 \|\nabla\omega^2\|_\infty \|\nabla\omega\|_2 + \left\|\frac{1}{\omega^1\omega^2}\right\|_\infty \|b^2\|_\infty \|\omega\|_2 \|\nabla\omega^2\|_\infty \|\nabla\omega\|_2 + \kappa_2\|\omega^1+\omega^2\|_\infty \|\omega\|_2^2.$$

By the Young inequality and Sobolev embedding theorem, we obtain

$$\begin{aligned}\frac{d}{dt}\|\omega\|_2^2 + \mu^t_{\min}\|\nabla\omega\|_2^2 \leqslant \frac{C}{\mu^t_{\min}}\Big(&\|\omega^1\|_{2,2}^2\|v\|_2^2 + (\omega^t_{\min})^{-2}\|\omega^2\|_{3,2}^2\|b\|_2^2 \\ &+ \left(\|v^2\|_{2,2}^2 + (\omega^t_{\min})^{-4}\|b^2\|_{2,2}^2\|\omega^2\|_{3,2}^2 + \mu^t_{\min}\|\omega^1\|_{2,2}^2 + \mu^t_{\min}\|\omega^2\|_{2,2}\right)\|\omega\|_2^2\Big),\end{aligned} \tag{3.28}$$

where C depends only on Ω and κ_2. Finally, we test the equations for b^1 and b^2 by $b = b^1 - b^2$ and we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|b\|_2^2 + \left(\frac{b^1}{\omega^1}\nabla b, \nabla b\right) &= \left(b^1 v, \nabla b\right) + \left(b v^2, \nabla b\right) - \left(\frac{b}{\omega^1}\nabla b^2, \nabla b\right) + \left(\frac{b^2\omega}{\omega^1\omega^2}\nabla b^2, \nabla b\right) \\ &- \left(b^1\omega, b\right) - \left(b\omega^2, b\right) + \left(\frac{b^1}{\omega^1}\left|D(v^1)\right|^2 - \frac{b^2}{\omega^2}\left|D(v^2)\right|^2, b\right).\end{aligned}$$

We note that the last term on the right-hand side is equal to

$$\left(\frac{b^1}{\omega^1}D(v)D(v^1+v^2), b\right) + \left(\frac{b}{\omega^1}\left|D(v^2)\right|^2, b\right) - \left(\frac{b^2\omega}{\omega^1\omega^2}\left|D(v^2)\right|^2, b\right).$$

From the Hölder inequality and (3.21), (3.22) it follows that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|b\|_2^2 + \mu^t_{\min}\|\nabla b\|_2^2 &\leqslant \left\|b^1\right\|_\infty \|v\|_2 \|\nabla b\|_2 + \|b\|_2 \left\|v^2\right\|_\infty \|\nabla b\|_2 \\
&+ \left\|\frac{1}{\omega^1}\right\|_\infty \|b\|_2 \left\|\nabla b^2\right\|_\infty \|\nabla b\|_2 + \left\|\frac{1}{\omega^1\omega^2}\right\|_\infty \left\|b^2\right\|_\infty \|\omega\|_2 \left\|\nabla b^2\right\|_\infty \|\nabla b\|_2 \\
&+ \left\|b^1\right\|_\infty \|\omega\|_2 \|b\|_2 + \|b\|_2 \left\|\omega^2\right\|_\infty \|b\|_2 \\
&+ \left\|\frac{1}{\omega^1}\right\|_\infty \left\|b^1\right\|_\infty \|D(v)\|_2 \left\|D(v^1+v^2)\right\|_\infty \|b\|_2 \\
&+ \left\|\frac{1}{\omega^1}\right\|_\infty \|b\|_2^2 \|D(v^2)\|_\infty^2 + \left\|\frac{1}{\omega^1\omega^2}\right\|_\infty \left\|b^2\right\|_\infty \|\omega\|_2 \left\|D(v^2)\right\|_\infty^2 \|b\|_2 .
\end{aligned}$$

Applying the Young inequality and Sobolev embedding theorem yields

$$\begin{aligned}
\frac{d}{dt}\|b\|_2^2 + \mu^t_{\min}\|\nabla b\|_2^2 \leqslant \frac{C}{\mu^t_{\min}}\Big\{&\|b^1\|_{2,2}^2\|v\|_2^2 + \Big[\|v^2\|_{2,2}^2 + (\omega^t_{\min})^{-2}\|b^2\|_{3,2}^2 + \|\omega^2\|_{2,2}^2 \\
&+ (\omega^t_{\min})^{-2}(\mu^t_{\min} + \|b^1\|_{2,2}^2)(\|v^1\|_{3,2}^2 + \|v^2\|_{3,2}^2) + \mu^t_{\min}(\omega^t_{\min})^{-1}\|v^2\|_{3,2}^2\Big]\|b\|_2^2 \\
&+ \Big[(\omega^t_{\min})^{-4}\|b^2\|_{2,2}^2\|b^2\|_{3,2}^2 + \mu^t_{\min}(\omega^t_{\min})^{-2}\|b^2\|_{2,2}^2\|v^2\|_{3,2}^2\Big]\|\omega\|_2^2\Big\} \\
&+ \mu^t_{\min}\|D(v)\|_2^2 + C(1+\|\omega^2\|_{2,2})\|b\|_2^2 + C\|b^1\|_{2,2}^2\|\omega\|_2^2.
\end{aligned} \tag{3.29}$$

Summing up the inequalities (3.27)-(3.29), we obtain

$$\frac{d}{dt}\left(\|v\|_2^2 + \|\omega\|_2^2 + \|b\|_2^2\right) \leqslant h(t)\left(\|v\|_2^2 + \|\omega\|_2^2 + \|b\|_2^2\right),$$

with $h \in L^1(0,T)$, because (v^i, ω^i, b^i) belong to $L^\infty(0,T;H^2(\Omega)) \cap L^2(0,T;H^3(\Omega))$. As $v(0) = 0$, $\omega(0) = 0$, $b(0) = 0$ hold, by the Grönwall inequality we get that $v \equiv 0$, $\omega \equiv 0$ and $b \equiv 0$ on Ω^T and the proof is finished. □

Suppose that the assumptions of Theorem 3.2.1 hold. Then, by Theorem 2.1.1 there exists a regular, local in-time solution to the system (1)-(5), which belongs to $\mathcal{X}(T_0)$ for some positive T_0. From Proposition 3.3.2 it is unique solution in $\mathcal{X}(T_0)$. We will show that provided the smallness condition imposed on initial data (formulated in (3.16)), the solution exists on $[0,T)$. In particular, if (3.16) holds with $T = \infty$, then the solution is global, i.e. it belongs to $\mathcal{X}(\infty)$. Firstly, we denote

$$T^* = \sup\{t^* > 0 : \quad \text{system (1)-(5) has a solution } (v, \omega, b) \quad \text{in } \mathcal{X}(t^*)\}. \tag{3.30}$$

We note that $T^* \geqslant T_0 > 0$. By Proposition 3.3.2 there exists (v, ω, b) the unique solution of (1)-(5) in $\mathcal{X}(T^*)$, i.e. the following identities

$$(v_{,t}, w) - (v \otimes v, \nabla w) + \left(\frac{b}{\omega} D(v), D(w)\right) = 0 \quad \text{for} \quad w \in \dot{\mathcal{V}}^1_{\text{div}}, \tag{3.31}$$

$$(\omega_{,t}, z) - (\omega v, \nabla z) + \left(\frac{b}{\omega} \nabla \omega, \nabla z\right) = -\kappa_2(\omega^2, z) \quad \text{for} \quad z \in \mathcal{V}^1, \tag{3.32}$$

$$(b_{,t}, q) - (bv, \nabla q) + \left(\frac{b}{\omega} \nabla b, \nabla q\right) = -(b\omega, q) + \left(\frac{b}{\omega} |D(v)|^2, q\right) \quad \text{for} \quad q \in \mathcal{V}^1 \tag{3.33}$$

hold for a.a. $t \in (0, T^*)$, where $(\cdot, \cdot)$ denotes the inner product in $L^2(\Omega)$. By Proposition 3.3.1 functions ω and b satisfy

$$b(t, x) \geqslant b^t_{\min}, \quad \omega(t, x) \geqslant \omega^t_{\min}, \quad \omega(t, x) \leqslant \omega^t_{\max} \quad \text{for} \quad (x, t) \in \Omega^{T^*}. \tag{3.34}$$

We shall show that if the condition (3.16) holds for some T, then $T^* \geqslant T$. As it will be explained in the proof of Corollary 3.2.4.1, the condition (3.16) holds, provided the initial data are sufficiently small.

To prove the result we suppose that $T^* < T$ and we shall show that it leads to a contradiction. The idea of the proof is as follows: we shall show that under smallness assumption (3.16) we are able to obtain an estimate for solution in $H^2(\Omega)$ norm, which is uniform with respect to $t \in [0, T^*)$. Next, by applying Theorem 2.1.1 and Proposition 3.3.1 we will be able to extend the solution beyond T^* and this is a contradiction with the definition of T^*. Therefore, the key step in the proof is to get estimates in the H^2 norm for the solution (v, ω, b). First, we deal with the lower-order terms.

3.3.1. The lower order estimates

In this subsection we estimate the L^2-norm of v and next, the L^1-norm of b. The proof of the main theorem depends heavily on the decay estimates of these quantities. In the proposition below we consider all values of $\kappa_2 \in (0, \infty)$ to illustrate the influence of κ_2 for the available decay estimates. From this, we will see that $\kappa_2 = \frac{1}{2}$ seems to be a critical value.

Proposition 3.3.3. *For each $t \in [0, T^*)$ the following estimates hold:*

a)

$$\|v(t)\|_2^2 \leqslant \|v_0\|_2^2 \exp\left(-\frac{1}{C_p^2}\frac{b_{\min}}{\omega_{\max}^2\,(2\kappa_2 - 1)}\left((1+\kappa_2\omega_{\max}t)^{2-\frac{1}{\kappa_2}} - 1\right)\right) \quad \text{for } \kappa_2 \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right), \tag{3.35}$$

and

$$\|v(t)\|_2^2 \leqslant \|v_0\|_2^2 (1+\kappa_2\omega_{\max}t)^{-\frac{b_{\min}}{C_p^2\omega_{\max}^2\kappa_2}} \quad \text{for} \quad \kappa_2 = \frac{1}{2}, \tag{3.36}$$

b)

$$\|\omega(t)\|_2 \leqslant \|\omega_0\|_2 \quad \text{for } \kappa_2 \in (0, \infty), \tag{3.37}$$

c)

$$\|b(t)\|_1 + \frac{1}{2}\|v(t)\|_2^2 \leqslant \frac{\|b_0\|_1 + \frac{1}{2}\|v_0\|_2^2\left(1 + I_\infty\left(\kappa_2, \frac{\omega_{\min}}{\omega_{\max}}, \frac{b_{\min}}{(\omega_{\max})^2}\right)\right)}{(1+\kappa_2\omega_{\min}t)^{\frac{1}{\kappa_2}}} \quad \text{for } \kappa_2 \in \left(\frac{1}{2}, \infty\right), \tag{3.38}$$

d)

$$\frac{\|b_0\|_1 + \frac{1}{2}\|v_0\|_2^2}{(1+\kappa_2\omega_{\max}t)^{\frac{1}{\kappa_2}}} \leqslant \|b(t)\|_1 + \frac{1}{2}\|v(t)\|_2^2 \quad \text{for } \kappa_2 \in (0, \infty), \tag{3.39}$$

e)

$$\|b\|_1 + \frac{1}{2}\|v(t)\|_2^2 \leqslant \|b_0\|_1 + \frac{1}{2}\|v_0\|_2^2 \quad \text{for } \kappa_2 \in (0, \infty), \tag{3.40}$$

where I_∞ was defined in (3.5).

Proof of Proposition 3.3.3. **a)** We test the equation (3.31) by v and we get

$$\frac{1}{2}\frac{d}{dt}\|v\|_2^2 + \left(\frac{b}{\omega}D(v), D(v)\right) = 0 \quad \text{for} \quad t \in (0, T^*), \tag{3.41}$$

where we applied the condition $\operatorname{div} v = 0$. Using notation (3.6) and the estimate (3.34) we obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|_2^2 + \mu_{\min}^t \|D(v)\|_2^2 \leqslant 0 \quad \text{for} \quad t \in (0, T^*).$$

The mean value of components of v are zero. Thus, from the Poincaré inequality and the fact that $\|D(v)\|_2 = \frac{\sqrt{2}}{2}\|\nabla v\|_2$ we get

$$\frac{d}{dt}\|v\|_2^2 + \mu_{\min}^t \frac{1}{C_p^2}\|v\|_2^2 \leqslant 0 \quad \text{for} \quad t \in (0, T^*).$$

By applying (3.6) we may write explicitly

$$\frac{d}{dt}\|v(t)\|_2^2 + \frac{1}{C_p^2}\frac{b_{\min}}{\omega_{\max}}\left(1 + \kappa_2 \omega_{\max} t\right)^{1 - \frac{1}{\kappa_2}} \|v(t)\|_2^2 \leqslant 0 \quad \text{for} \quad t \in (0, T^*). \tag{3.42}$$

Multiplying by an appropriate exponential function gives

$$\frac{d}{dt}\left[\|v(t)\|_2^2 \exp\left(\frac{b_{\min}}{C_p^2 \omega_{\max}^2 \kappa_2 (2 - \frac{1}{\kappa_2})}\left(1 + \kappa_2 \omega_{\max} t\right)^{2 - \frac{1}{\kappa_2}}\right)\right] \leqslant 0 \quad \text{for} \quad t \in (0, T^*).$$

After integration we obtain (3.35). Similarly we derive (3.36).

b) If we test the equation (3.32) by $z = \omega$, then after integration by parts and using (3.34) we get

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|_2^2 \leqslant 0 \quad \text{for } t \in (0, T^*).$$

Thus we have (3.37).

c) We now proceed to estimate for b. We can not obtain any pointwise estimate from above for b. However, we are able to estimate the L^1-norm of b. Indeed, we test the equation (3.33) by $q \equiv 1$ and we get

$$(b_{,t}, 1) = -(b\omega, 1) + \left(\frac{b}{\omega}|D(v)|^2, 1\right).$$

The positivity of b follows from (3.1), (3.4) and (3.34), so we get

$$\frac{d}{dt}\|b\|_1 = -(b\omega, 1) + \left(\frac{b}{\omega}|D(v)|^2, 1\right).$$

We note that the term $\left(\frac{b}{\omega}|D(v)|^2, 1\right)$ is equal to $\left(\frac{b}{\omega}D(v), D(v)\right)$. Therefore, we can use the equation (3.41) and we obtain

$$\frac{d}{dt}\|b\|_1 = -\left(b\omega, 1\right) - \frac{1}{2}\frac{d}{dt}\|v\|_2^2. \tag{3.43}$$

From (3.3) and (3.34) we may estimate ω from below and we obtain

$$\frac{d}{dt}\|b\|_1 \leqslant -\frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t}\|b\|_1 - \frac{1}{2}\frac{d}{dt}\|v\|_2^2. \tag{3.44}$$

Multiplying both sides by $e^{\int_0^t \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}\tau}d\tau}$ yields

$$\frac{d}{dt}\left(\|b\|_1 e^{\int_0^t \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}\tau}d\tau}\right) \leqslant -\frac{1}{2}\frac{d}{dt}\|v\|_2^2 e^{\int_0^t \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}\tau}d\tau}.$$

Integrating from 0 to t gives

$$\|b\|_1 e^{\int_0^t \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}\tau}d\tau} \leqslant \|b_0\|_1 - \frac{1}{2}\int_0^t \frac{d}{d\tau}\|v(\tau)\|_2^2 e^{\int_0^\tau \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}s}ds}d\tau.$$

After integrating by parts we get

$$\begin{aligned}\|b\|_1 e^{\int_0^t \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}\tau}d\tau} \leqslant \|b_0\|_1 &- \left[\frac{1}{2}\|v(\tau)\|_2^2 e^{\int_0^\tau \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}s}ds}\right]_{\tau=0}^{\tau=t} \\ &+ \frac{1}{2}\int_0^t \|v(\tau)\|_2^2 e^{\int_0^\tau \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}s}ds}\frac{\omega_{\min}}{1+\kappa_2\omega_{\min}\tau}d\tau.\end{aligned}$$

Thus we get

$$\begin{aligned}\|b\|_1 + \frac{1}{2}\|v\|_2^2 \leqslant &\left(\|b_0\|_1 + \frac{1}{2}\|v_0\|_2^2\right) e^{-\int_0^t \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}\tau}d\tau} \\ &+ \frac{1}{2}e^{-\int_0^t \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}\tau}d\tau}\int_0^t \|v(\tau)\|_2^2 e^{\int_0^\tau \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}s}ds}\frac{\omega_{\min}}{1+\kappa_2\omega_{\min}\tau}d\tau.\end{aligned}$$

We note that

$$\int_0^t \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}\tau}d\tau = \ln\left(1+\kappa_2\omega_{\min}t\right)^{\frac{1}{\kappa_2}},$$

so we obtain

$$\|b\|_1 + \frac{1}{2}\|v\|_2^2 \leqslant \frac{\|b_0\|_1 + \frac{1}{2}\|v_0\|_2^2}{(1+\kappa_2\omega_{\min}t)^{\frac{1}{\kappa_2}}} + \frac{1}{2}\frac{1}{(1+\kappa_2\omega_{\min}t)^{\frac{1}{\kappa_2}}}\int_0^t \|v(\tau)\|_2^2 \frac{\omega_{\min}}{(1+\kappa_2\omega_{\min}\tau)^{1-\frac{1}{\kappa_2}}}d\tau.$$

Using (3.35) implies

$$\|b\|_1 + \frac{1}{2}\|v\|_2^2 \leqslant \frac{\|b_0\|_1 + \frac{1}{2}\|v_0\|_2^2}{(1+\kappa_2\omega_{\min}t)^{\frac{1}{\kappa_2}}} + \frac{\frac{1}{2}\|v_0\|_2^2}{(1+\kappa_2\omega_{\min}t)^{\frac{1}{\kappa_2}}} I_t(\kappa_2,\omega_{\min},\omega_{\max},b_{\min}),$$

where

$$I_t(\kappa_2,\omega_{\min},\omega_{\max},b_{\min}) = \int_0^t \exp\left(-\frac{b_{\min}\left((1+\kappa_2\omega_{\max}\tau)^{2-\frac{1}{\kappa_2}}-1\right)}{C_p^2\omega_{\max}^2(2\kappa_2-1)}\right)\frac{\omega_{\min}}{(1+\kappa_2\omega_{\min}\tau)^{1-\frac{1}{\kappa_2}}}d\tau. \tag{3.45}$$

Now, we shall derive an estimate of I_t. Depending on the value of κ_2, we obtain different types of estimates. Firstly, we focus on the case $\kappa_2 \geqslant 1$. From (3.21) we have

$$\frac{\omega_{\min}}{(1+\kappa_2\omega_{\min}\tau)^{1-\frac{1}{\kappa_2}}} = \frac{(\omega_{\min})^{\frac{1}{\kappa_2}}(\omega_{\min})^{1-\frac{1}{\kappa_2}}}{(1+\kappa_2\omega_{\min}\tau)^{1-\frac{1}{\kappa_2}}} \leqslant \frac{(\omega_{\min})^{\frac{1}{\kappa_2}}(\omega_{\max})^{1-\frac{1}{\kappa_2}}}{(1+\kappa_2\omega_{\max}\tau)^{1-\frac{1}{\kappa_2}}}$$

and thus

$$I_t(\kappa_2,\omega_{\min},\omega_{\max},b_{\min}) \leqslant \omega_{\max}\left(\frac{\omega_{\min}}{\omega_{\max}}\right)^{\frac{1}{\kappa_2}}\int_0^t \exp\left(-\frac{b_{\min}\left((1+\kappa_2\omega_{\max}\tau)^{2-\frac{1}{\kappa_2}}-1\right)}{C_p^2\omega_{\max}^2(2\kappa_2-1)}\right)\frac{d\tau}{(1+\kappa_2\omega_{\max}\tau)^{1-\frac{1}{\kappa_2}}}. \tag{3.46}$$

Now, we change variables $s = 1+\kappa_2\omega_{\max}\tau$ and get

$$I_t(\kappa_2,\omega_{\min},\omega_{\max},b_{\min}) \leqslant \frac{1}{\kappa_2}\left(\frac{\omega_{\min}}{\omega_{\max}}\right)^{\frac{1}{\kappa_2}}\exp\left(\frac{b_{\min}}{C_p^2\omega_{\max}^2(2\kappa_2-1)}\right)\int_1^\infty \exp\left(-\frac{b_{\min}s^{2-\frac{1}{\kappa_2}}}{C_p^2\omega_{\max}^2(2\kappa_2-1)}\right)s^{\frac{1}{\kappa_2}-1}ds.$$

Next, the change of variables $y = \frac{b_{\min} s^{2-\frac{1}{\kappa_2}}}{C_p^2 \omega_{\max}^2 (2\kappa_2 - 1)}$ leads to the estimate

$$
\begin{aligned}
&I_t(\kappa_2, \omega_{\min}, \omega_{\max}, b_{\min}) \\
&\leqslant \left(\frac{\omega_{\min}}{\omega_{\max}}\right)^{\frac{1}{\kappa_2}} \exp\left(\frac{b_{\min}}{C_p^2 \omega_{\max}^2 (2\kappa_2 - 1)}\right) \left(\frac{C_p^2 (\omega_{\max})^2 (2\kappa_2 - 1)}{b_{\min}}\right)^{\frac{1}{2\kappa_2 - 1}} \Gamma\left(\frac{2\kappa_2}{2\kappa_2 - 1}\right).
\end{aligned}
$$

Therefore, in the case of $\kappa_2 \geqslant 1$ we obtain

$$
\begin{aligned}
&\|b\|_1 + \frac{1}{2}\|v\|_2^2 \\
&\leqslant \frac{\|b_0\|_1 + \frac{1}{2}\|v_0\|_2^2 \left(1 + \Gamma\left(\frac{2\kappa_2}{2\kappa_2 - 1}\right) \left(\frac{\omega_{\min}}{\omega_{\max}}\right)^{\frac{1}{\kappa_2}} \left(\frac{C_p^2 (\omega_{\max})^2 (2\kappa_2 - 1)}{b_{\min}} \exp\left(\frac{b_{\min}}{C_p^2 \omega_{\max}^2}\right)\right)^{\frac{1}{2\kappa_2 - 1}}\right)}{(1 + \kappa_2 \omega_{\min} t)^{\frac{1}{\kappa_2}}}.
\end{aligned}
$$

Hence (3.38) holds for $\kappa_2 \geqslant 1$. Now, if we assume that $\kappa_2 \in \left(\frac{1}{2}, 1\right)$, then we have

$$
\frac{1}{(1 + \kappa_2 \omega_{\min} \tau)^{1 - \frac{1}{\kappa_2}}} \leqslant \frac{1}{(1 + \kappa_2 \omega_{\max} \tau)^{1 - \frac{1}{\kappa_2}}}
$$

and from (3.45) we obtain

$$
\begin{aligned}
&I_t(\kappa_2, \omega_{\min}, \omega_{\max}, b_{\min}) \\
&\leqslant \omega_{\min} \int_0^t \exp\left(-\frac{b_{\min}\left((1 + \kappa_2 \omega_{\max} \tau)^{2 - \frac{1}{\kappa_2}} - 1\right)}{C_p^2 \omega_{\max}^2 (2\kappa_2 - 1)}\right) \frac{d\tau}{(1 + \kappa_2 \omega_{\max} \tau)^{1 - \frac{1}{\kappa_2}}}.
\end{aligned}
\tag{3.47}
$$

Proceeding as before we obtain

$$
\begin{aligned}
&\|b\|_1 + \frac{1}{2}\|v\|_2^2 \\
&\leqslant \frac{\|b_0\|_1 + \frac{1}{2}\|v_0\|_2^2 \left(1 + \Gamma\left(\frac{2\kappa_2}{2\kappa_2 - 1}\right) \frac{\omega_{\min}}{\omega_{\max}} \left(\frac{C_p^2 (\omega_{\max})^2 (2\kappa_2 - 1)}{b_{\min}} \exp\left(\frac{b_{\min}}{C_p^2 \omega_{\max}^2}\right)\right)^{\frac{1}{2\kappa_2 - 1}}\right)}{(1 + \kappa_2 \omega_{\min} t)^{\frac{1}{\kappa_2}}}.
\end{aligned}
$$

Hence, (3.38) also holds for $\kappa_2 \in \left(\frac{1}{2}, 1\right)$.

d) Now, we shall obtain (3.39) - the estimate from below. Firstly, we note that from (3.21) and (3.43) we have

$$
\frac{d}{dt}\left(\|b\|_1 + \frac{1}{2}\|v\|_2^2\right) \geqslant -\frac{\omega_{\max}}{1 + \kappa_2 \omega_{\max} t}\|b\|_1.
$$

Therefore

$$\frac{d}{dt}\ln\left(\|b\|_1+\frac{1}{2}\|v\|_2^2\right)\geqslant-\frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}.$$

After integrating the both sides from 0 to t we obtain

$$\ln\left(\frac{\|b\|_1+\frac{1}{2}\|v\|_2^2}{\|b_0\|_1+\frac{1}{2}\|v_0\|_2^2}\right)\geqslant-\frac{1}{\kappa_2}\ln\left(1+\kappa_2\omega_{\max}t\right),$$

so the inequality (3.39) is proved.

e) The estimate (3.40) is a direct consequence of (3.44).

□

3.3.2. Higher order estimates

In this section we will obtain estimates for $\|\Delta v(t)\|_2$, $\|\Delta\omega(t)\|_2$ and $\|\Delta b(t)\|_2$. Having these estimates and results of the previous section we will be able to control the H^2 norm. From (3.31)-(3.33) we get

$$(v_{,t},\Delta^2 w)-(v\otimes v,\nabla\Delta^2 w)+\left(\frac{b}{\omega}D(v),D(\Delta^2 w)\right)=0, \tag{3.48}$$

$$(\omega_{,t},\Delta^2 z)-(\omega v,\nabla\Delta^2 z)+\left(\frac{b}{\omega}\nabla\omega,\nabla\Delta^2 z\right)=-\kappa_2(\omega^2,\Delta^2 z), \tag{3.49}$$

$$(b_{,t},\Delta^2 q)-(bv,\nabla\Delta^2 q)+\left(\frac{b}{\omega}\nabla b,\nabla\Delta^2 q\right)=-(b\omega,\Delta^2 q)+\left(\frac{b}{\omega}\left|D(v)\right|^2,\Delta^2 q\right) \tag{3.50}$$

for a.a. $t\in(0,T^*)$, where the test functions are such that $\Delta^2 w\in\dot{\mathcal{V}}^1_{\mathrm{div}}$, $\Delta^2 z\in\mathcal{V}^1$ and $\Delta^2 q\in\mathcal{V}^1$. Integrating by parts and using the condition $\operatorname{div} v=0$, we obtain

$$\langle\Delta v_{,t},\Delta w\rangle-(\Delta\left(v\otimes v\right),\nabla\Delta w)+\left(\Delta\left(\frac{b}{\omega}D(v)\right),D(\Delta w)\right)=0, \tag{3.51}$$

$$\begin{aligned}\langle\Delta\omega_{,t},\Delta z\rangle-(v\nabla^2\omega,\nabla\Delta z)-(\nabla\omega\nabla v,\nabla\Delta z)+&\left(\Delta\left(\frac{b}{\omega}\nabla\omega\right),\nabla\Delta z\right)\\&=-\kappa_2(\Delta\left(\omega^2\right),\Delta z),\end{aligned} \tag{3.52}$$

$$\begin{aligned}\langle\Delta b_{,t},\Delta q\rangle-(v\nabla^2 b,\nabla\Delta q)-(\nabla b\nabla\omega,\nabla\Delta q)+\left(\Delta\left(\frac{b}{\omega}\nabla b\right),\nabla\Delta q\right)&\\=-(\Delta\left(b\omega\right),\Delta q)-\left(\nabla\left(\frac{b}{\omega}\left|D(v)\right|^2\right),\Delta\nabla q\right),&\end{aligned} \tag{3.53}$$

for a.a. $t \in (0, T^*)$, where $\langle \cdot, \cdot \rangle$ denotes duality pairing between $\mathcal{V}^1(\Omega)$ and $(\mathcal{V}^1)^*$. The density argument and regularity of (v, ω, b) allow us to test the system (3.51)-(3.53) by the solution. Thus we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta v\|_2^2 - \left(\Delta\left(v \otimes v\right), \nabla\Delta v\right) + \left(\Delta\left(\frac{b}{\omega}D(v)\right), D(\Delta v)\right) = 0, \tag{3.54}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta\omega\|_2^2 - (v\nabla^2\omega, \nabla\Delta\omega) - (\nabla\omega\nabla v, \nabla\Delta\omega)& \\ + \left(\Delta\left(\frac{b}{\omega}\nabla\omega\right), \nabla\Delta\omega\right) &= -\kappa_2(\Delta\left(\omega^2\right), \Delta\omega),\end{aligned} \tag{3.55}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta b\|_2^2 - (v\nabla^2 b, \nabla\Delta b) - (\nabla b\nabla v, \nabla\Delta b) + \left(\Delta\left(\frac{b}{\omega}\nabla b\right), \nabla\Delta b\right)& \\ = -(\Delta\left(b\omega\right), \Delta b) - \left(\nabla\left(\frac{b}{\omega}\left|D(v)\right|^2\right), \nabla\Delta b\right)&\end{aligned} \tag{3.56}$$

for a.a. $t \in (0, T^*)$. In the above equations, some terms are similar and can be treated in the same way. To simplify further calculations let us analyse these terms first. One of them has the following form

$$\left(\Delta\left(\frac{b}{\omega}\nabla f\right), \nabla\Delta f\right).$$

In this case, we may write

$$\begin{aligned}\left(\Delta\left(\frac{b}{\omega}\nabla f\right), \nabla\Delta f\right) &= \left(\frac{b}{\omega}\nabla\Delta f, \nabla\Delta f\right) + 2\left(\nabla^2 f \cdot \nabla\left(\frac{b}{\omega}\right), \nabla\Delta f\right) \\ + \left(\Delta\left(\frac{b}{\omega}\right)\nabla f, \nabla\Delta f\right) &= \left(\frac{b}{\omega}\nabla\Delta f, \nabla\Delta f\right) + 2\left(\frac{1}{\omega}\nabla^2 f \cdot \nabla b, \nabla\Delta f\right) \\ - 2\left(\frac{b}{\omega^2}\nabla^2 f \cdot \nabla\omega, \nabla\Delta f\right) &+ \left(\frac{\Delta b}{\omega}\nabla f, \nabla\Delta f\right) - 2\left(\frac{(\nabla b \cdot \nabla\omega)}{\omega^2}\nabla f, \nabla\Delta f\right) \\ - \left(\frac{b}{\omega^2}\Delta\omega\nabla f, \nabla\Delta f\right) &+ 2\left(\frac{b}{\omega^3}\left|\nabla\omega\right|^2\nabla f, \nabla\Delta f\right).\end{aligned} \tag{3.57}$$

On the right-hand side, we can control the sign only of the first term. Therefore, to simplify future calculations we define $W(f)$ using the last six expressions, i.e.

$$\left(\Delta\left(\frac{b}{\omega}\nabla f\right), \nabla\Delta f\right) = \left(\frac{b}{\omega}\nabla\Delta f, \nabla\Delta f\right) + W(f). \tag{3.58}$$

Similarly we define $\widetilde{W}(v)$

$$\left(\Delta\left(\frac{b}{\omega}D(v)\right), D(\Delta v)\right) = \left(\frac{b}{\omega}D(\Delta v), D(\Delta v)\right) + \widetilde{W}(v). \tag{3.59}$$

Using this notation the system (3.54)-(3.56) may be written in the following way

$$\frac{1}{2}\frac{d}{dt}\|\Delta v\|_2^2 + \left(\frac{b}{\omega}D(\Delta v), D(\Delta v)\right) = (\Delta\,(v \otimes v)\,, \nabla \Delta v) - \widetilde{W}(v) \tag{3.60}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta \omega\|_2^2 + \left(\frac{b}{\omega}\nabla\Delta\omega, \nabla\Delta\omega\right) &= -\kappa_2(\Delta(\omega^2), \Delta\omega) + (v\nabla^2\omega, \nabla\Delta\omega)\\ &\quad +(\nabla\omega\nabla v, \nabla\Delta\omega) - W(\omega),\end{aligned} \tag{3.61}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta b\|_2^2 + \left(\frac{b}{\omega}\nabla\Delta b, \nabla\Delta b\right) &= (v\nabla^2 b, \nabla\Delta b) + (\nabla b\nabla v, \nabla\Delta b)\\ -(\Delta\,(b\omega)\,, \Delta b) &- \left(\nabla\left(\frac{b}{\omega}\,|D(v)|^2\right), \nabla\Delta b\right) - W(b).\end{aligned} \tag{3.62}$$

We recall that by applying (3.6) and (3.34) we get the bound from below

$$\mu_{\min}^t \leqslant \frac{b}{\omega}. \tag{3.63}$$

Thus, from (3.60) we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta v\|_2^2 + \mu_{\min}^t\|D(\Delta v)\|_2^2 \leqslant 2(\Delta v \otimes v, \nabla\Delta v) + 2(\nabla v \otimes \nabla v, \nabla\Delta v) - \widetilde{W}(v). \tag{3.64}$$

To estimate the right-hand side we use the Hölder inequality and we get

$$2(\Delta v \otimes v, \nabla\Delta v) + 2(\nabla v \otimes \nabla v, \nabla\Delta v) \leqslant 2\|v\|_3\|\Delta v\|_6\,\|\nabla\Delta v\|_2 + 2\|\nabla v\|_4^2\,\|\nabla\Delta v\|_2\,.$$

Then, after applying the Sobolev inequalities and Gagliardo-Nierenberg inequality (1.6) we get

$$2(\Delta v \otimes v, \nabla\Delta v) + 2(\nabla v \otimes \nabla v, \nabla\Delta v) \leqslant C(\|v\|_3 + \|\nabla v\|_2)\|\nabla^3 v\|_2^2,$$

where C depends only on Ω. Using the interpolating inequality

$$\|v\|_3 \leqslant C\,\|v\|_2^{\frac{1}{2}}\,\|\nabla v\|_2^{\frac{1}{2}}\,,$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta v\|_2^2 + \mu_{\min}^t\|D(\Delta v)\|_2^2 \leqslant C\left(\|v\|_2^{\frac{1}{2}}\,\|\nabla v\|_2^{\frac{1}{2}} + \|\nabla v\|_2\right)\|\nabla^3 v\|_2^2 - \widetilde{W}(v). \tag{3.65}$$

Now we focus our attention on the equation (3.61). After applying (3.63) we get

$$\frac{1}{2}\frac{d}{dt}\|\Delta\omega\|_2^2 + \mu_{\min}^t\|\nabla\Delta\omega\|_2^2 \leqslant (v\nabla^2\omega, \nabla\Delta\omega) + (\nabla\omega\nabla v, \nabla\Delta\omega)$$
$$- 2\kappa_2(|\nabla\omega|^2, \Delta\omega) - W(\omega),$$

where we used the nonnegativity of $2\kappa_2(\omega\Delta\omega, \Delta\omega)$. By the Hölder inequality we have

$$\frac{1}{2}\frac{d}{dt}\|\Delta\omega\|_2^2 + \mu_{\min}^t\|\nabla\Delta\omega\|_2^2 \leqslant \|v\|_3\|\nabla^2\omega\|_6\|\nabla\Delta\omega\|_2 + 2\|\nabla\omega\|_4\|\nabla v\|_4\|\nabla\Delta\omega\|_2$$
$$+ 2\kappa_2\|\nabla\omega\|_{\frac{6}{5}}\|\nabla\omega\|_\infty\|\Delta\omega\|_6 - W(\omega).$$

Applying the estimate (1.12) to the term $\|\nabla\omega\|_\infty$ and (1.7) to term $\|v\|_3$ yields

$$\frac{1}{2}\frac{d}{dt}\|\Delta\omega\|_2^2 + \mu_{\min}^t\|\nabla\Delta\omega\|_2^2 \leqslant C\Big(\|v\|_2^{\frac{1}{2}}\|\nabla v\|_2^{\frac{1}{2}} + \kappa_2\|\nabla\omega\|_{\frac{6}{5}}\Big)\|\nabla^3\omega\|_2^2$$
$$+ 2\|\nabla\omega\|_4\|\nabla v\|_4\|\nabla\Delta\omega\|_2 - W(\omega).$$

Using inequality (1.10) gives

$$\frac{1}{2}\frac{d}{dt}\|\Delta\omega\|_2^2 + \mu_{\min}^t\|\nabla\Delta\omega\|_2^2 \leqslant C\Big(\|v\|_2^{\frac{1}{2}}\|\nabla v\|_2^{\frac{1}{2}} + \kappa_2\|\nabla\omega\|_{\frac{6}{5}}\Big)\|\nabla^3\omega\|_2^2$$
$$+ C\|\nabla\omega\|_{\frac{3}{2}}^{\frac{1}{2}}\|\nabla v\|_{\frac{3}{2}}^{\frac{1}{2}}\|\nabla^3\omega\|_2^{\frac{3}{2}}\|\nabla^3 v\|_2^{\frac{1}{2}} - W(\omega),$$

where C depends only on Ω. So finally, after applying the Young inequality with exponents $(\frac{4}{3}, 4)$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta\omega\|_2^2 + \mu_{\min}^t\|\nabla\Delta\omega\|_2^2 \leqslant C\Big(\|v\|_2^{\frac{1}{2}}\|\nabla v\|_2^{\frac{1}{2}} + \kappa_2\|\nabla\omega\|_{\frac{6}{5}} + \|\nabla\omega\|_{\frac{3}{2}}^{\frac{1}{2}}\|\nabla v\|_{\frac{3}{2}}^{\frac{1}{2}}\Big)$$
$$\cdot\left(\|\nabla^3\omega\|_2^2 + \|\nabla^3 v\|_2^2\right) - W(\omega). \tag{3.66}$$

Now, let us turn our attention to equation (3.62). We integrate by parts

$$-(\Delta(b\omega), \Delta b) = -(\omega\Delta b, \Delta b) - 2(\nabla\omega\nabla b, \Delta b) - (b\Delta\omega, \Delta b),$$

$$\left(\nabla\left(\frac{b}{\omega}|D(v)|^2\right), \nabla\Delta b\right) = \left(\frac{\nabla b}{\omega}|D(v)|^2, \nabla\Delta b\right) - \left(\frac{b\nabla\omega}{\omega^2}|D(v)|^2, \nabla\Delta b\right)$$
$$+ 2\left(\frac{b}{\omega}D(v)\nabla D(v), \nabla\Delta b\right).$$

Using the above calculations we may write (3.62) in the following form

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta b\|_2^2 + \left(\frac{b}{\omega}\nabla\Delta b, \nabla\Delta b\right) &= (v\nabla^2 b, \nabla\Delta b) + (\nabla b\nabla v, \nabla\Delta b) - (\omega\Delta b, \Delta b)\\ &- 2(\nabla\omega\nabla b, \Delta b) - (b\Delta\omega, \Delta b) - \left(\frac{\nabla b}{\omega}|D(v)|^2, \nabla\Delta b\right) + \left(\frac{b\nabla\omega}{\omega^2}|D(v)|^2, \nabla\Delta b\right)\\ &- 2\left(\frac{b}{\omega}D(v)\nabla D(v), \nabla\Delta b\right) - W(b).\end{aligned}$$

The third term on the right-hand side is non-positive. Hence, using (3.63), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta b\|_2^2 + \mu_{\min}^t\|\nabla\Delta b\|_2^2 &\leqslant (\nabla b\nabla v, \nabla\Delta b) + (v\nabla^2 b, \nabla\Delta b)\\ &- 2(\nabla\omega\nabla b, \Delta b) - (b\Delta\omega, \Delta b) - \left(\frac{\nabla b}{\omega}|D(v)|^2, \nabla\Delta b\right)\\ &+ \left(\frac{b\nabla\omega}{\omega^2}|D(v)|^2, \nabla\Delta b\right) - 2\left(\frac{b}{\omega}D(v)\nabla D(v), \nabla\Delta b\right) - W(b).\end{aligned} \tag{3.67}$$

From the Hölder inequality we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta b\|_2^2 + \mu_{\min}^t\|\nabla\Delta b\|_2^2 &\leqslant \|\nabla b\|_4\|\nabla v\|_4\left\|\nabla\Delta b\right\|_2 + \|v\|_3\|\nabla^2 b\|_6\left\|\nabla\Delta b\right\|_2\\ &+ 2\left\|\nabla\omega\right\|_{\frac{6}{5}}\left\|\nabla b\right\|_\infty\left\|\Delta b\right\|_6 + \|b\|_{\frac{3}{2}}\left\|\Delta\omega\right\|_6\|\Delta b\|_6 + \left\|\frac{1}{\omega}\right\|_\infty\|\nabla b\|_6\|D(v)\|_6^2\left\|\nabla\Delta b\right\|_2\\ &+ \left\|\frac{1}{\omega}\right\|_\infty^2\|b\|_\infty\left\|\nabla\omega\right\|_6\|D(v)\|_6^2\left\|\nabla\Delta b\right\|_2 + 2\left\|\frac{1}{\omega}\right\|_\infty\|b\|_\infty\left\|\nabla D(v)\right\|_6\|D(v)\|_3\left\|\nabla\Delta b\right\|_2 - W(b).\end{aligned}$$

Now, we estimate the right-hand side by applying the Gagliardo-Nirenberg inequalities

by (1.10) : $\qquad \|\nabla b\|_4\|\nabla v\|_4\left\|\nabla\Delta b\right\|_2 \leqslant c\left\|\nabla b\right\|_{\frac{3}{2}}^{\frac{1}{2}}\left\|\nabla v\right\|_{\frac{3}{2}}^{\frac{1}{2}}\left\|\nabla^3 b\right\|_2^{\frac{3}{2}}\left\|\nabla^3 v\right\|_2^{\frac{1}{2}},$

by (1.7), (1.8) : $\qquad \|v\|_3\|\nabla^2 b\|_6\left\|\nabla\Delta b\right\|_2 \leqslant c\left\|\nabla v\right\|_2^{\frac{1}{2}}\left\|v\right\|_2^{\frac{1}{2}}\left\|\nabla^3 b\right\|_2^2,$

by (1.8), (1.12) : $\qquad \left\|\nabla\omega\right\|_{\frac{6}{5}}\left\|\nabla b\right\|_\infty\left\|\Delta b\right\|_6 \leqslant c\left\|\nabla\omega\right\|_{\frac{6}{5}}\left\|\nabla^3 b\right\|_2^2,$

by (1.13), (1.8) : $\qquad \|b\|_{\frac{3}{2}}\left\|\Delta\omega\right\|_6\|\Delta b\|_6 \leqslant c(\|\nabla b\|_{\frac{3}{2}}^{\frac{1}{2}}\left\|b\right\|_1^{\frac{1}{2}} + \|b\|_1)\left\|\nabla^3\omega\right\|_2\left\|\nabla^3 b\right\|_2,$

by (3.34), (1.8), (1.9) : $\qquad \left\|\frac{1}{\omega}\right\|_\infty\|\nabla b\|_6\|D(v)\|_6^2\left\|\nabla\Delta b\right\|_2$

$$\leqslant c(\omega_{\min}^t)^{-1}\left\|\nabla^2 b\right\|_2\left\|\nabla v\right\|_2\left\|\nabla^3 v\right\|_2\left\|\nabla^3 b\right\|_2,$$

by (3.34), (1.11), (1.8), (1.9) : $\qquad \left\|\frac{1}{\omega}\right\|_\infty^2\|b\|_\infty\left\|\nabla\omega\right\|_6\|D(v)\|_6^2\left\|\nabla\Delta b\right\|_2$

$$\leqslant c(\omega^t_{\min})^{-2}(\left\|\nabla^2 b\right\|_2 + \|b\|_1)\left\|\nabla^2\omega\right\|_2\|\nabla v\|_2\left\|\nabla^3 v\right\|_2\left\|\nabla^3 b\right\|_2,$$

by (3.34), (1.11), (1.8), (1.7) : $\left\|\frac{1}{\omega}\right\|_\infty \|b\|_\infty \|\nabla D(v)\|_6 \|D(v)\|_3 \|\nabla\Delta b\|_2$

$$\leqslant c(\omega^t_{\min})^{-1}(\left\|\nabla^2 b\right\|_2 + \|b\|_1)\left\|\nabla^3 v\right\|_2\|\nabla v\|_2^{\frac{1}{2}}\left\|\nabla^2 v\right\|_2^{\frac{1}{2}}\left\|\nabla^3 b\right\|_2,$$

where c depends only on Ω. Thus, if we apply the Young inequality to separate the norms of the third-order derivatives, then we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta b\|_2^2 + \mu^t_{\min}\|\nabla\Delta b\|_2^2 \leqslant c\Big(&\|\nabla b\|_{\frac{3}{2}}^{\frac{1}{2}}\|\nabla v\|_{\frac{3}{2}}^{\frac{1}{2}} + \|\nabla v\|_2^{\frac{1}{2}}\|v\|_2^{\frac{1}{2}} + \|\nabla\omega\|_{\frac{6}{5}} + \|\nabla b\|_{\frac{3}{2}}^{\frac{1}{2}}\|b\|_1^{\frac{1}{2}}\\ &+ \|b\|_1 + (\omega^t_{\min})^{-1}\left\|\nabla^2 b\right\|_2\|\nabla v\|_2 + (\omega^t_{\min})^{-2}\left(\left\|\nabla^2 b\right\|_2 + \|b\|_1\right)\left\|\nabla^2\omega\right\|_2\|\nabla v\|_2\\ &+ (\omega^t_{\min})^{-1}\left(\left\|\nabla^2 b\right\|_2 + \|b\|_1\right)\|\nabla v\|_2^{\frac{1}{2}}\left\|\nabla^2 v\right\|_2^{\frac{1}{2}}\Big)\cdot\Big(\|\nabla^3 v\|_2^2 + \|\nabla^3\omega\|_2^2 + \|\nabla^3 b\|_2^2\Big)\\ &- W(b),\end{aligned} \tag{3.68}$$

where c depends only on Ω. We note that after integration by parts we get $\left\|\nabla^2 f\right\|_2 = \|\Delta f\|_2$ for $f \in \mathcal{V}^2$ and $2\|D(\Delta v)\|_2^2 = \|\nabla^3 v\|_2^2$ (see (2.31), (2.32)). Hence, summing the inequalities (3.65), (3.66) and (3.68), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\Delta v\|_2^2 + \|\Delta\omega\|_2^2 + \|\Delta b\|_2^2\right) + \mu^t_{\min}\left(\|\nabla\Delta v\|_2^2 + \|\nabla\Delta\omega\|_2^2 + \|\nabla\Delta b\|_2^2\right)\\ &\quad\leqslant C\Big(\|v\|_2^{\frac{1}{2}}\|\nabla v\|_2^{\frac{1}{2}} + \|\nabla v\|_2 + \|\nabla\omega\|_{\frac{6}{5}} + \|\nabla\omega\|_{\frac{3}{2}}^{\frac{1}{2}}\|\nabla v\|_{\frac{3}{2}}^{\frac{1}{2}} + \|\nabla b\|_{\frac{3}{2}}^{\frac{1}{2}}\|\nabla v\|_{\frac{3}{2}}^{\frac{1}{2}}\\ &\qquad + \|\nabla b\|_{\frac{3}{2}}^{\frac{1}{2}}\|b\|_1^{\frac{1}{2}} + \|b\|_1 + (\omega^t_{\min})^{-1}\left\|\nabla^2 b\right\|_2\|\nabla v\|_2 + (\omega^t_{\min})^{-2}\left\|\nabla^2 b\right\|_2\left\|\nabla^2\omega\right\|_2\|\nabla v\|_2\\ &\qquad + \frac{\|b\|_1}{(\omega^t_{\min})^2}\left\|\nabla^2\omega\right\|_2\|\nabla v\|_2 + (\omega^t_{\min})^{-1}\left\|\nabla^2 b\right\|_2\|\nabla v\|_2^{\frac{1}{2}}\left\|\nabla^2 v\right\|_2^{\frac{1}{2}} + \frac{\|b\|_1}{\omega^t_{\min}}\|\nabla v\|_2^{\frac{1}{2}}\left\|\nabla^2 v\right\|_2^{\frac{1}{2}}\Big)\\ &\qquad\cdot\Big(\|\nabla\Delta v\|_2^2 + \|\nabla\Delta\omega\|_2^2 + \|\nabla\Delta b\|_2^2\Big) - \widetilde{W}(v) - W(\omega) - W(b),\end{aligned}$$

where C depends only on κ_2 and Ω. Before we estimate the last three terms we will introduce the following notation

$$\begin{aligned}X_0(t) &:= \|v(t)\|_2^2 + \|b(t)\|_1^2,\\ X_1(t) &:= \|\nabla v(t)\|_2^2 + \|\nabla\omega(t)\|_2^2 + \|\nabla b(t)\|_2^2,\\ X_2(t) &:= \|\Delta v(t)\|_2^2 + \|\Delta\omega(t)\|_2^2 + \|\Delta b(t)\|_2^2,\\ X_3(t) &:= \|\nabla\Delta v(t)\|_2^2 + \|\nabla\Delta\omega(t)\|_2^2 + \|\nabla\Delta b(t)\|_2^2.\end{aligned} \tag{3.69}$$

After using the Hölder inequality we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}X_2+\mu^t_{\min}X_3 \leqslant C\Big(&X_0^{\frac14}X_1^{\frac14}+X_1^{\frac12}+\|b\|_1+(\omega^t_{\min})^{-1}X_1^{\frac12}X_2^{\frac12}+(\omega^t_{\min})^{-2}X_1^{\frac12}X_2\\
&+\frac{\|b\|_1}{(\omega^t_{\min})^2}X_1^{\frac12}X_2^{\frac12}+(\omega^t_{\min})^{-1}X_1^{\frac14}X_2^{\frac34}+\frac{\|b\|_1}{\omega^t_{\min}}X_1^{\frac14}X_2^{\frac14}\Big)\cdot X_3-\widetilde{W}(v)-W(\omega)-W(b),\end{aligned} \tag{3.70}$$

where C depends only on κ_2 and Ω. Now, we need to estimate terms $\widetilde{W}(v)$, $W(\omega)$, $W(b)$, which were defined by (3.57)-(3.59). In each case the estimates are similar. Thus, we consider $W(f)$ for general $f\in\mathcal{V}^3$. In this case, we have

$$\begin{aligned}|W(f)|\leqslant{}& 2\left\|\frac{1}{\omega}\right\|_\infty\|\nabla b\|_3\|\nabla^2 f\|_6\,\|\nabla\Delta f\|_2+2\left\|\frac{1}{\omega}\right\|_\infty^2\|b\|_\infty\,\|\nabla\omega\|_3\|\nabla^2 f\|_6\,\|\nabla\Delta f\|_2\\
&+\left\|\frac{1}{\omega}\right\|_\infty\|\Delta b\|_6\|\nabla f\|_3\,\|\nabla\Delta f\|_2+2\left\|\frac{1}{\omega}\right\|_\infty^2\|\nabla b\|_6\|\nabla\omega\|_6\|\nabla f\|_6\,\|\nabla\Delta f\|_2\\
&+\left\|\frac{1}{\omega}\right\|_\infty^2\|b\|_\infty\,\|\Delta\omega\|_6\|\nabla f\|_3\,\|\nabla\Delta f\|_2+2\left\|\frac{1}{\omega}\right\|_\infty^3\|b\|_\infty\,\|\nabla\omega\|_6^2\|\nabla f\|_6\,\|\nabla\Delta f\|_2\,.\end{aligned}$$

As before, we use (3.34) and (1.7)-(1.13) and we have

$$\begin{aligned}|W(f)|\leqslant\frac{c}{\omega^t_{\min}}\Big(&\|\nabla b\|_2^{\frac12}\|\Delta b\|_2^{\frac12}\left\|\nabla^3 f\right\|_2+(\omega^t_{\min})^{-1}\left(\|\Delta b\|_2+\|b\|_1\right)\|\nabla\omega\|_2^{\frac12}\|\Delta\omega\|_2^{\frac12}\left\|\nabla^3 f\right\|_2\\
&+\|\nabla f\|_2^{\frac12}\|\Delta f\|_2^{\frac12}\|\nabla\Delta b\|_2+(\omega^t_{\min})^{-1}\|\nabla b\|_2^{\frac12}\|\nabla\omega\|_2^{\frac12}\|\Delta f\|_2\|\nabla\Delta\omega\|_2^{\frac12}\|\nabla\Delta b\|_2^{\frac12}\\
&+(\omega^t_{\min})^{-1}\left(\|\Delta b\|_2+\|b\|_1\right)\|\nabla f\|_2^{\frac12}\|\Delta f\|_2^{\frac12}\|\nabla\Delta\omega\|_2\\
&+(\omega^t_{\min})^{-2}\left(\|\Delta b\|_2+\|b\|_1\right)\|\nabla\omega\|_2\|\Delta f\|_2\|\nabla\Delta\omega\|_2\Big)\|\nabla\Delta f\|_2\,,\end{aligned}$$

where c depends only on Ω. We obtain an analogous estimate for $\widetilde{W}(v)$. Then, using notation (3.69), we obtain

$$\begin{aligned}|\widetilde{W}(v)|+|W(\omega)|+|W(b)|\leqslant\frac{c}{\omega^t_{\min}}\Big(&X_1^{\frac14}X_2^{\frac14}+(\omega^t_{\min})^{-1}X_1^{\frac14}X_2^{\frac34}+\frac{\|b\|_1}{\omega^t_{\min}}X_1^{\frac14}X_2^{\frac14}+X_1^{\frac14}X_2^{\frac14}\\
&+(\omega^t_{\min})^{-1}X_1^{\frac12}X_2^{\frac12}+(\omega^t_{\min})^{-1}X_1^{\frac14}X_2^{\frac34}+\frac{\|b\|_1}{\omega^t_{\min}}X_1^{\frac14}X_2^{\frac14}\\
&+\frac{1}{(\omega^t_{\min})^2}X_1^{\frac12}X_2+\frac{\|b\|_1}{(\omega^t_{\min})^2}X_1^{\frac12}X_2^{\frac12}\Big)\cdot X_3,\end{aligned}$$

where c is as earlier. We simplify further

$$
\begin{aligned}
|\widetilde{W}(v)| + |W(\omega)| + |W(b)| \leqslant \frac{c}{(\omega^t_{\min})^2}\Big(&\left(\omega^t_{\min} + \|b\|_1\right) X_1^{\frac14} X_2^{\frac14} + \left(1 + \frac{\|b\|_1}{\omega^t_{\min}}\right) X_1^{\frac12} X_2^{\frac12} \\
&+ X_1^{\frac14} X_2^{\frac34} + (\omega^t_{\min})^{-1} X_1^{\frac12} X_2 \Big) \cdot X_3
\end{aligned}
\tag{3.71}
$$

and c depends only on Ω. Using this estimate in (3.70) we get

$$
\begin{aligned}
\frac12 \frac{d}{dt} X_2 + \mu^t_{\min} X_3 \leqslant C \Bigg(& X_0^{\frac14} X_1^{\frac14} + X_1^{\frac12} + \|b\|_1 + \left(\frac{1}{\omega^t_{\min}} + \frac{\|b\|_1}{\omega^t_{\min}} + \frac{\|b\|_1}{(\omega^t_{\min})^2}\right) X_1^{\frac14} X_2^{\frac14} \\
&+ \left(\frac{1}{\omega^t_{\min}} + \frac{1}{(\omega^t_{\min})^2} + \frac{\|b\|_1}{(\omega^t_{\min})^2} + \frac{\|b\|_1}{(\omega^t_{\min})^3}\right) X_1^{\frac12} X_2^{\frac12} \\
&+ \left(\frac{1}{\omega^t_{\min}} + \frac{1}{(\omega^t_{\min})^2}\right) X_1^{\frac14} X_2^{\frac34} + \left(\frac{1}{(\omega^t_{\min})^2} + \frac{1}{(\omega^t_{\min})^3}\right) X_1^{\frac12} X_2 \Bigg) \cdot X_3,
\end{aligned}
\tag{3.72}
$$

where $C = C(\Omega, \kappa_2)$. After applying the Poincaré inequality we obtain $X_1 \leqslant C_p^2 X_2$, so we may simplify further

$$
\begin{aligned}
&\frac12 \frac{d}{dt} X_2 + \mu^t_{\min} X_3 \leqslant C \Big(X_0^{\frac14} X_2^{\frac14} + \|b\|_1 + \left(1 + \frac{1}{\omega^t_{\min}} + \frac{\|b\|_1}{\omega^t_{\min}} + \frac{\|b\|_1}{(\omega^t_{\min})^2}\right) X_2^{\frac12} \\
&+ \left(\frac{1}{\omega^t_{\min}} + \frac{1}{(\omega^t_{\min})^2} + \frac{\|b\|_1}{(\omega^t_{\min})^2} + \frac{\|b\|_1}{(\omega^t_{\min})^3}\right) X_2 + \left(\frac{1}{(\omega^t_{\min})^2} + \frac{1}{(\omega^t_{\min})^3}\right) X_2^{\frac32} \Big) \cdot X_3.
\end{aligned}
\tag{3.73}
$$

By (3.6) and (3.38) we have $\|b(t)\|_1 \leqslant b_{\max}(t)$. Hence, using (3.8), (3.35), (3.38) and (3.69) we get

$$
X_0^{\frac14}(t) \leqslant \left(\|v_0\|_2^2 \exp\left(- \frac{b_{\min}\left((1 + \kappa_2 \omega_{\max} t)^{2 - \frac{1}{\kappa_2}} - 1 \right)}{C_p^2 \omega_{\max}^2 (2\kappa_2 - 1)} \right) + b^2_{\max}(t) \right)^{\frac14} \equiv A(t)
$$

and we obtain

$$
X_0^{\frac14} X_2^{\frac14} + \|b\|_1 \leqslant A(t) X_2^{\frac14} + b_{\max}(t).
$$

Applying this inequality in (3.73) yields

$$
\frac{d}{dt} X_2 + 2\mu^t_{\min} X_3 \leqslant C_{\Omega,\kappa_2} \Big(b_{\max}(t) + A(t) X_2^{\frac14} + B(t) X_2^{\frac12} + C(t) X_2 + D(t) X_2^{\frac32} \Big) \cdot X_3, \tag{3.74}
$$

where C_{Ω,κ_2} depends only on Ω, κ_2 and we used the notation (3.9)-(3.11). We denote

$$Z(t) = \Big(b_{\max}(t) + A(t)X_2^{\frac{1}{4}} + B(t)X_2^{\frac{1}{2}} + C(t)X_2 + D(t)X_2^{\frac{3}{2}}\Big). \tag{3.75}$$

Thus, the inequality (3.74) may be written in the following form

$$\frac{d}{dt}X_2(t) + \left(\mu^t_{\min} - C_{\Omega,\kappa_2}Z(t)\right)X_3(t) \leqslant -\mu^t_{\min}X_3(t).$$

Using the Poincaré inequality implies

$$\frac{d}{dt}X_2(t) + \left(\mu^t_{\min} - C_{\Omega,\kappa_2}Z(t)\right)X_3(t) \leqslant -\frac{\mu^t_{\min}}{C_p^2}X_2(t). \tag{3.76}$$

By definition (3.7) and (3.69) we have $Y_2(0) = X_2(0)$. Hence, using (3.12) and (3.75) we get $Z_0(0) = Z(0)$. Next, by assumption (3.16) we have

$$\frac{b_{\min}}{\omega_{\max}} - C_{\Omega,\kappa_2}Z_0(0) > 0,$$

so we have

$$\frac{b_{\min}}{\omega_{\max}} - C_{\Omega,\kappa_2}Z(0) > 0.$$

We note that $(v,\omega,b) \in L^2([0,T^*);H^3(\Omega))$ and $(v_{,t},\omega_{,t},b_{,t}) \in L^2([0,T^*);H^1(\Omega))$. Hence, we have $X_2 \in C([0,T^*))$. Therefore, there are two possibilities:

$$\forall t \in [0,T^*) \quad \mu^t_{\min} - C_{\Omega,\kappa_2}Z(t) > 0 \quad \text{or} \quad \exists t^* \in (0,T^*) \quad \mu^{t^*}_{\min} - Z(t^*) = 0.$$

In the first case, the inequality (3.76) gives a uniform estimate

$$\|\Delta v(t)\|_2^2 + \|\Delta\omega(t)\|_2^2 + \|\Delta b(t)\|_2^2 \leqslant \|\Delta v_0\|_2^2 + \|\Delta\omega_0\|_2^2 + \|\Delta b_0\|_2^2 \quad \text{for} \quad t \in [0,T^*). \tag{3.77}$$

By (3.35)-(3.40) we have

$$\|v(t)\|_2 \leqslant \|v_0\|_2\,, \quad \|\omega(t)\|_2 \leqslant \|\omega_0\|_2\,,$$

$$\|b(t)\|_2 \leqslant c(\left\|\nabla^2 b(t)\right\|_2 + \|b(t)\|_1) \leqslant c(\left\|\nabla^2 b(t)\right\|_2 + \|b_0\|_1 + \frac{1}{2}\|v_0\|_2^2)$$

for $t \in [0, T^*)$, where $c = c(\Omega)$. These estimates together with (3.77) give

$$\|v(t)\|_{2,2}^2 + \|\omega(t)\|_{2,2}^2 + \|b(t)\|_{2,2}^2 \leqslant c \left(\|v_0\|_{2,2}^2 + \|\omega_0\|_{2,2}^2 + \|b_0\|_{2,2}^2 \right) \tag{3.78}$$

for $t \in [0, T^*)$, where c depends only on Ω. We denote the right-hand side of (3.78) by δ. We set $K = \{(\omega_{\min}^t, \omega_{\max}^t, b_{\min}^t) : \ t \in [0, T^*]\}$. Then K is the compact subset of $\{(a, b, c) : \ 0 < a \leqslant b, \ 0 < c\}$ and by Theorem 2.1.1 there exists $t_{K,\delta}^*$ such that the problem (1)-(5) with the initial condition $(v(t), \omega(t), b(t))$ can be extended to the interval $[t, t + t_{K,\delta}^*)$, where t is arbitrary in $[0, T^*)$. For $t > T^* - t_{K,\delta}^*$ we obtain the contradiction with definition of T^* (see (3.30)).

In the second case, using the continuity of $[0, T^*) \ni t \mapsto \mu_{\min}^t - C_{\Omega,\kappa_2} Z(t)$ we may assume that $t^* \in (0, T^*)$ is the first point with this property, i.e. $\mu_{\min}^t - C_{\Omega,\kappa_2} Z(t) > 0$ for $t \in [0, t^*)$ and $\mu_{\min}^{t^*} - Z(t^*) = 0$. Then, from (3.76) it follows

$$\frac{d}{dt} X_2(t) \leqslant -\frac{1}{C_p^2} \mu_{\min}^t X_2(t) \quad \text{for} \quad t \in (0, t^*).$$

Using (3.6) we may write

$$\frac{d}{dt} X_2(t) \leqslant -\frac{1}{C_p^2} \frac{b_{\min}}{\omega_{\max}} \left(1 + \kappa_2 \omega_{\max} t\right)^{1 - 1/\kappa_2} X_2(t) \quad \text{for} \quad t \in (0, t^*).$$

Thus, after multiplying by an appropriate exponential function we obtain the bound

$$X_2(t) \leqslant X_2(0) \exp\left(-\frac{1}{C_p^2} \frac{b_{\min}}{(2\kappa_2 - 1)\omega_{\max}^2} \left((1 + \kappa_2 \omega_{\max} t)^{2 - 1/\kappa_2} - 1 \right) \right) \quad \text{for} \quad t \in (0, t^*).$$

By definition (3.7), the above inequality means $X_2(t) \leqslant Y_2(t)$ for $t \in [0, t^*)$. Hence, we get $X_2(t^*) \leqslant Y_2(t^*)$. Using the definition (3.12) and (3.75), we deduce that $Z(t^*) \leqslant Z_0(t^*)$ and then

$$0 = \mu_{\min}^{t^*} - C_{\Omega,\kappa_2} Z(t^*) \geqslant \mu_{\min}^{t^*} - C_{\Omega,\kappa_2} Z_0(t^*) > 0,$$

so we get a contradiction with the assumption (3.16). Thus, we obtain that $T^* \geqslant T$ and the theorem 3.2.1 is proved.

It remains to prove Corollary 3.2.4.1.

3.4. Proof of Corollary 3.2.4.1

We shall show that the condition (3.16) is satisfied for $T = \infty$. Firstly, for $\kappa_2 \geqslant 1$ we note that from (3.17) it follows

$$\frac{b_{\min}}{\omega_{\max}} > 2C_{\Omega,\kappa_2}\left(\|b_0\|_1 + \frac{1}{2}\|v_0\|_2^2\left(1 + I_\infty\left(\kappa_2, \frac{\omega_{\min}}{\omega_{\max}}, \frac{b_{\min}}{(\omega_{\max})^2}\right)\right)\right)(1+\kappa_2\omega_{\min}t)^{-1}$$

for $t \geqslant 0$. Hence, after multiplying the both sides by $(1+\kappa_2\omega_{\min}t)^{1-\frac{1}{\kappa_2}}$ we get

$$\mu^t_{\min} \geqslant \frac{b_{\min}}{\omega_{\max}}(1+\kappa_2\omega_{\min}t)^{1-\frac{1}{\kappa_2}} > 2C_{\Omega,\kappa_2}b_{\max}(t). \tag{3.79}$$

For $\kappa_2 \in \left(\frac{1}{2}, 1\right)$ from (3.18) it follows

$$\frac{b_{\min}}{\omega_{\max}}(1+\kappa_2\omega_{\max}t)$$
$$> 2C_{\Omega,\kappa_2}\left(\frac{\omega_{\max}}{\omega_{\min}}\right)^{\frac{1}{\kappa_2}}\left(\|b_0\|_1 + \frac{1}{2}\|v_0\|_2^2\left(1 + I_\infty\left(\kappa_2, \frac{\omega_{\min}}{\omega_{\max}}, \frac{b_{\min}}{(\omega_{\max})^2}\right)\right)\right)$$

for $t \geqslant 0$. Hence, after multiplying the both sides by $(1+\kappa_2\omega_{\max}t)^{-\frac{1}{\kappa_2}}$ we get

$$\mu^t_{\min} > 2C_{\Omega,\kappa_2}b_{\max}(t)\left(\frac{\omega_{\max}}{\omega_{\min}} \cdot \frac{1+\kappa_2\omega_{\min}t}{1+\kappa_2\omega_{\max}t}\right)^{\frac{1}{\kappa_2}}.$$

We note that the function $\frac{1+\kappa_2\omega_{\min}t}{1+\kappa_2\omega_{\max}t}$ is decreasing and strictly greater than $\frac{\omega_{\min}}{\omega_{\max}}$, so we have

$$\mu^t_{\min} > 2C_{\Omega,\kappa_2}b_{\max}(t). \tag{3.80}$$

Next, we shall show that a_0 is finite. Recall that $\kappa_2 > \frac{1}{2}$ and then by (3.4), (3.8) we deduce that $(1+\kappa_2\omega_{\max}t)^{\frac{1}{\kappa_2}-1}A(t)$ decays at infinity as $(1+\kappa_2\omega_{\max}t)^{\frac{1}{2\kappa_2}-1}$. Thus, the expression $(1+\kappa_2\omega_{\max}t)^{\frac{1}{\kappa_2}-1}A(t)$ is uniformly bounded on $[0,\infty)$. Furthermore, the remaining terms in the definition of a_0 can be estimated by expressions of the form $(1+\kappa_2\omega_{\max}t)^\alpha Y_2^\beta(t)$, where $\alpha \leqslant 3$ and $\beta > 0$. We recall that the function $Y_2(t)$ decays exponentially, therefore a_0 is finite.

Finally, by (3.19) we get $\frac{b_{\min}}{\omega_{\max}} > a_0 Y_2^{\frac{1}{4}}(t)$ for $t \in [0,\infty)$. Thus, using the definition of a_0 we obtain

$$\frac{b_{\min}}{\omega_{\max}} > 2C_{\Omega,\kappa_2}(1+\kappa_2\omega_{\max}t)^{\frac{1}{\kappa_2}-1}\left(A(t)Y_2^{\frac{1}{4}}(t) + B(t)Y_2^{\frac{1}{2}} + C(t)Y_2(t) + D(t)Y_2^{\frac{3}{2}}(t)\right)$$

for $t \in [0, \infty)$, so we get

$$\mu^t_{\min} > 2C_{\Omega,\kappa_2} \left(A(t)Y_2^{\frac{1}{4}}(t) + B(t)Y_2^{\frac{1}{2}} + C(t)Y_2(t) + D(t)Y_2^{\frac{3}{2}}(t) \right). \tag{3.81}$$

Summing (3.79) or (3.80), (3.81) and using definition (3.12) we get $2\mu^t_{\min} > 2C_{\Omega,\kappa_2} Z_0(t)$. Hence, the condition (3.16) holds for $T = \infty$.

Chapter 4

Existence and uniqueness of local in time solutions for $H^s(\mathbb{T}^d)$ initial data

In this section, we will show the existence of the local in-time solutions for initial data from $H^s(\mathbb{T}^d)$ and $s > \frac{d}{2}$. Presented methodology heavily relies on the results of Chapter A. Moreover, the restriction on s follows from interpolation inequalities, Lemma 1.3.3 and the Grönwall inequality. The results were published in [29] in the form of a preprint.

4.1. Notation and main result

Assume that $\Omega = \mathbb{T}^d$, $T > 0$ and $\Omega^T = \Omega \times (0, T)$. We shall consider problem (1)-(5) in Ω^T. Constants $\nu_0, \kappa_1, \ldots, \kappa_4$ are positive. For simplicity, we assume further that all constants except of κ_2 are equal to one. The reason is that the constant κ_2 plays an important role in a priori estimates.

We shall show the local-in-time existence of a regular solution of problem (1)-(5) under some assumption imposed on the initial data. Namely, suppose that $v_0 \in H^s_{\text{div}}(\mathbb{T}^d)$, ω_0, $b_0 \in H^s(\mathbb{T}^d)$, where $s > \frac{d}{2}$ and that there exist positive numbers $b_{\min}$, $\omega_{\min}$, $\omega_{\max}$ such that

$$0 < b_{\min} \leqslant b_0(x), \tag{4.1}$$

$$0 < \omega_{\min} \leqslant \omega_0(x) \leqslant \omega_{\max} \tag{4.2}$$

on $\mathbb{T}^d$. Additionally, based on the introduced bound we define

$$\begin{gathered} b^t_{\min} = \frac{b_{\min}}{(1+\kappa_2\omega_{\max}t)^{\frac{1}{\kappa_2}}}, \quad \omega^t_{\min} = \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t}, \\ \omega^t_{\max} = \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}, \quad \mu^t_{\min} = \frac{1}{4}\frac{b^t_{\min}}{\omega^t_{\max}}. \end{gathered} \tag{4.3}$$

Now, we introduce the notion of a solution to the system (1)-(5). Let $s > \frac{d}{2}$. We say that triple (v, ω, b) such that

$$v \in C([0,t^*), H^s_{\text{div}}(\mathbb{T}^d)) \cap L^2(0,t^*, H^{s+1}_{\text{div}}(\mathbb{T}^d)) \cap W^{1,2}(0,t^*, H^{s-1}_{\text{div}}(\mathbb{T}^d)), \tag{4.4}$$

$$(\omega, b) \in \left(C([0,t^*), H^s\left(\mathbb{T}^d\right)) \cap L^2(0,t^*, H^{s+1}\left(\mathbb{T}^d\right)) \cap W^{1,2}(0,t^*, H^{s-1}(\mathbb{T}^d))\right)^2 \tag{4.5}$$

is a solution to (1)-(5) on time interval $[0, T)$ if

$$(v_{,t}, w) - (v \otimes v, \nabla w) + (\mu D(v), D(w)) = 0 \quad \text{for} \quad w \in H^1_{\text{div}}(\mathbb{T}^d), \tag{4.6}$$

$$(\omega_{,t}, z) - (\omega v, \nabla z) + (\mu \nabla \omega, \nabla z) = -\kappa_2(\omega^2, z) \quad \text{for} \quad z \in H^1(\mathbb{T}^d), \tag{4.7}$$

$$(b_{,t}, q) - (bv, \nabla q) + (\mu \nabla b, \nabla q) = -(b\omega, q) + (\mu |D(v)|^2, q) \quad \text{for} \quad q \in H^1(\mathbb{T}^d), \tag{4.8}$$

holds for a.a. $t \in (0, T)$, $\mu = \frac{b}{\omega}$ and (5) holds. Recall that $D(v)$ denotes the symmetric part of ∇v and $(\cdot, \cdot)$ is the inner product in $L^2(\Omega)$.

Now we can formulate existence and uniqueness results.

Theorem 4.1.1. *Let $d \in \mathbb{N}_{\geqslant 2}$, $s > d/2$. Let $(v_0, \omega_0, b_0) \in \left(H^s\left(\mathbb{T}^d\right)\right)^{d+2}$ be such that* $\operatorname{div} v_0 = 0$, $\min_{x \in \mathbb{T}^d} b_0(x) > 0$ *and* $\min_{x \in \mathbb{T}^d} \omega_0(x) > 0$. *Additionally, let time $T > 0$ be such that*

$$(1 - 2^{-\beta+1})\left(1 + \|v_0, \omega_0, b_0\|^2_{H^s}\right)^{-\beta+1} = (\beta - 1) \int_0^T C(b_{\min}, \omega_{\min}, s, \tau) d\tau,$$

where $C(\omega_{\min}, b_{\min}, s, \tau)$ is a rational function which is finite for $\tau \geqslant 0$ (see (4.45)) and $\beta = \beta(s) > 1$. Then, there exists $t^* > T$ such that the system (4.6)-(4.8) has a unique solution (v, ω, b) on $[0, t^*)$.

Theorem 4.1.2. *Let $d \in \mathbb{N}_{\geqslant 2}$, $s > d/2$. Let $(v_0, \omega_0, b_0) \in \left(H^s\left(\mathbb{T}^d\right)\right)^{d+2}$ be such that* $\operatorname{div} v_0 = 0$, $\min_{x \in \mathbb{T}^d} b_0(x) > 0$ *and* $\min_{x \in \mathbb{T}^d} \omega_0(x) > 0$. *Let $T > 0$ and (v_i, ω_i, b_i) be two solutions to system (4.6)-(4.8). Then*

$$v^1 = v^2, \qquad \omega^1 = \omega^2, \qquad b^1 = b^2 \qquad \forall (t, x) \in [0, T] \times \mathbb{T}^d.$$

4.2. Proof of Theorem 4.1.1

The proof will be divided into several steps to better present and simplify the used methodology.

4.2.1. Definitions of auxiliary functions

To define an approximate problem we have to introduce a few auxiliary functions. For fixed $t > 0$ we denote by $\Psi_t = \Psi_t(x)$ a smooth, non-decreasing function such that

$$\Psi_t(x) = \begin{cases} \frac{1}{2}b^t_{\min} & \text{for} \quad x < \frac{1}{2}b^t_{\min} \\ x & \text{for} \quad x \geqslant b^t_{\min} \end{cases}, \tag{4.9}$$

where $b^t_{\min}$ is defined by (4.3). We assume that the function Ψ_t also satisfies

$$|\Psi_t^{(k)}(x)| \leqslant c_0 (b^t_{\min})^{1-k} \quad \forall k \in \{1, \dots, \lceil s \rceil + 1\}, \tag{4.10}$$

where, c_0 is a constant independent of x and t (see Section 1.4 for details). We also need a smooth, non-decreasing function Φ_t such that

$$\Phi_t(x) = \begin{cases} \frac{1}{2}\omega^t_{\min} & \text{for} \quad x < \frac{1}{2}\omega^t_{\min} \\ x & \text{for} \quad x \in [\omega^t_{\min}, \omega^t_{\max}] \\ 2\omega^t_{\max} & \text{for} \quad x > 2\omega^t_{\max} \end{cases}. \tag{4.11}$$

We assume that this function additionally satisfy

$$|\Phi_t^{(k)}(x)| \leqslant c_0 (\omega^t_{\min})^{1-k} \quad \forall k \in \{1, \dots, \lceil s \rceil + 1\}, \tag{4.12}$$

for some constant c_0 (see Section 1.4 for details).

4.2.2. Approximated system

To obtain an approximate system we will follow the procedure used in [50], [49]. Let us define the operator P_n in the following way

$$P_n f(x) = \sum_{|k|<n} f_k e^{2\pi i k x}, \qquad f_k = \int_{\mathbb{T}^d} f(x) e^{-2\pi i k x} dx. \tag{4.13}$$

In later parts we will require some properties of the P_n operator. First, it is obvious that

$$P_n P_n = P_n,$$

which follows from the orthogonality in $L^2(\mathbb{T}^d)$ of the functions $\{e^{2\pi ikx}\}_{k\in\mathbb{Z}^d}$. Let us define $C(n,d) = \sum_{|k|<n} 1$ and observe that for $m \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$ we have

$$\begin{aligned}
\left\| \frac{\partial^m (P_n f)}{\partial x_{i_1} \dots \partial x_{i_m}} \right\|_p &= (2\pi)^m \left\| \sum_{|k|<n} k_{i_1} \dots k_{i_m} f_k e^{2\pi i k \cdot} \right\|_p \leqslant (2\pi)^m \sum_{|k|<n} |k_{i_1} \dots k_{i_m}| |f_k| \left\| e^{2\pi i k \cdot} \right\|_p \\
&\leqslant (2\pi n)^m \sum_{|k|<n} |f_k| \leqslant (2\pi n)^m \sqrt{C(n,d)} \left(\sum_{|k|<n} |f_k|^2 \right)^{\frac{1}{2}} \\
&= (2\pi n)^m \sqrt{C(n,d)} \left\| P_n f \right\|_2 .
\end{aligned}$$

Thus for $m \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$ we have

$$\left\| P_n f \right\|_{W^{m,p}(\mathbb{T}^d)} \leqslant C(n,m,d) \left\| P_n f \right\|_2 . \tag{4.14}$$

The obtained result is not surprising and could be justified based on the equivalence of norms in finite-dimensional spaces. Also, we can easily check that the order of differentiation and P_n, when sequentially applied to function $f : \mathbb{T}^d \to \mathbb{C}$, is interchangeable

$$\begin{aligned}
\left(P_n \frac{\partial f}{\partial x_i} \right)(x) &= \sum_{|k|<n} \int_{\mathbb{T}^d} \frac{\partial f(x')}{\partial x_i'} e^{-2\pi i k x'} dx' e^{2\pi i k x} = \sum_{|k|<n} 2\pi i k_i \int_{\mathbb{T}^d} f(x') e^{-2\pi i k x'} dx' e^{2\pi i k x} \\
&= \frac{\partial}{\partial x_i} \left(\sum_{|k|<n} \int_{\mathbb{T}^d} f(x') e^{-2\pi i k x'} dx' e^{2\pi i k x} \right) = \left(\frac{\partial}{\partial x_i} P_n f \right)(x).
\end{aligned}$$

Thus it is clear that the following relations hold:

$$P_n \operatorname{div} f = \operatorname{div} P_n f, \qquad P_n \nabla f = \nabla P_n f, \qquad P_n \Delta f = \Delta P_n f. \tag{4.15}$$

Before we define an ODE system we have to solve (for $\alpha^n_{k,j}$, γ^n_k, η^n_k), we introduce the following functions

$$\begin{aligned} v_{n,j}(x,t) &= \sum_{|k|<n} \alpha^n_{k,j}(t)e^{2\pi ikx}, \qquad \omega_n(x,t) = \sum_{|k|<n} \beta^n_k(t)e^{2\pi ikx}, \\ b_n(x,t) &= \sum_{|k|<n} \gamma^n_k(t)e^{2\pi ikx}, \qquad p_n(x,t) = \sum_{|k|<n} \eta^n_k(t)e^{2\pi ikx}. \end{aligned} \tag{4.16}$$

Thanks to the orthonormality of the basis $\{e^{2\pi ikx}\}_{k\in\mathbb{Z}^d}$ in $L^2(\mathbb{T}^d)$ we have

$$\|v_n(t)\|_2^2 = \sum_{|k|<n} |\alpha^n_{k,j}(t)|^2, \quad \|\omega_n(t)\|_2^2 = \sum_{|k|<n} |\beta^n_k(t)|^2, \quad \|b_n(t)\|_2^2 = \sum_{|k|<n} |\gamma^n_k(t)|^2. \tag{4.17}$$

Additionally, we define the function $\overline{\mu_n}$ is the following way:

$$\overline{\mu_n} = \frac{\Psi_t(\mathrm{Re}(b_n))}{\Phi_t(\mathrm{Re}(\omega_n))}. \tag{4.18}$$

The modification was introduced to guarantee positive signs of diffusive terms. Moreover, the presence of $\mathrm{Re}(\cdot)$ in the definition of $\overline{\mu_n}$ allows us to deal with possibly complex-valued solutions. Now we consider the following system of equations

$$\partial_t\alpha^n_{k,j} = \Big(\left(-P_n\left(v_n\cdot\nabla v_n\right) + \mathrm{div}\left(P_n\left(\overline{\mu_n}D(v_n)\right)\right) + \nabla p_n\right)_j, e^{-2\pi ik\cdot}\Big), \tag{4.19}$$

$$\partial_t\beta^n_k = \left(-P_n\left(v_n\cdot\nabla\omega_n\right) + \mathrm{div}\left(P_n\left(\overline{\mu_n}\nabla\omega_n\right)\right) - \kappa_2 P_n\left(\omega_n^2\right), e^{-2\pi ik\cdot}\right), \tag{4.20}$$

$$\partial_t\gamma^n_k = \left(-P_n\left(v_n\cdot\nabla b_n\right) + \mathrm{div}\left(P_n\left(\overline{\mu_n}\nabla b_n\right)\right) - P_n\left(b_n\omega_n\right) + P_n\left(\overline{\mu_n}|Dv_n|^2\right), e^{-2\pi ik\cdot}\right) \tag{4.21}$$

complemented by the initial conditions

$$\alpha^n_{k,j}(0) = \left(v_{0,j}, e^{-2\pi ik\cdot}\right), \quad \beta^n_k(0) = \left(\omega_0, e^{-2\pi ik\cdot}\right), \quad \gamma^n_k(0) = \left(b_0, e^{-2\pi ik\cdot}\right) \tag{4.22}$$

and the following equation from which the pressure is calculated

$$-\Delta p_n = \mathrm{div}\Big[P_n\left(v_n\cdot\nabla v_n\right) - \mathrm{div}\left(P_n\left(\overline{\mu_n}D(v_n)\right)\right)\Big], \qquad \int_{\mathbb{T}^d} p_n(x)dx = 0. \tag{4.23}$$

The introduced system of equations can be represented in the following form

$$\frac{d}{dt}\begin{bmatrix} \left(\alpha^n_{k,j}\right)_{\substack{k=1,\dots,n\\ j=1,\dots,d}} \\ \left(\beta^n_k\right)_{k=1,\dots,n} \\ \left(\gamma^n_k\right)_{k=1,\dots,n} \end{bmatrix} = F\left(\left(\alpha^n_{k,j}\right)_{\substack{k=1,\dots,n\\ j=1,\dots,d}}, \left(\beta^n_k\right)_{k=1,\dots,n}, \left(\gamma^n_k\right)_{k=1,\dots,n}, t\right). \tag{4.24}$$

To show the existence of a solution of system (4.19) - (4.23) we will show that the right-hand side is locally Lipschitz continuous with respect to $\alpha^n_{k,j}$, β^n_k, γ^n_k, so basically, we need to estimate

$$|F(\alpha^{n,2}_{k,j}, \beta^{n,2}_k, \gamma^{n,2}_k, t) - F(\alpha^{n,1}_{k,j}, \beta^{n,1}_k, \gamma^{n,1}_k, t)|.$$

To this end, we introduce the following functions

$$v^m_{n,j}(x) = \sum_{|k|<n} \alpha^{n,m}_{k,j} e^{2\pi ikx}, \qquad \omega^m_n(x) = \sum_{|k|<n} \beta^{n,m}_k e^{2\pi ikx}, \qquad b^m_n(x) = \sum_{|k|<n} \gamma^{n,m}_k e^{2\pi ikx}. \tag{4.25}$$

We also introduce

$$\overline{\mu^m_n} = \frac{\Psi_t(\mathrm{Re}(b^m_n))}{\Phi_t(\mathrm{Re}(\omega^m_n))}. \tag{4.26}$$

Additionally, functions p^m_n are calculated with the help of the system (4.23), with a natural substitution of functions on the right-hand side: $v_{n,j} \to v^m_{n,j}$, $\omega_n \to \omega^m_n$, $b_n \to b^m_n$.
We start checking the local Lipschitz continuity by considering the term of the right-hand side of (4.19) involving the pressure term. We see that

$$\left|\left((\nabla p^2_n)_j, e^{-2\pi ik\cdot}\right) - \left((\nabla p^1_n)_j, e^{-2\pi ik\cdot}\right)\right| \leqslant \left\|e^{-2\pi ik\cdot}\right\|_2 \left\|\nabla\left(p^2_n - p^1_n\right)\right\|_2 \leqslant \left\|\nabla\left(p^2_n - p^1_n\right)\right\|_2.$$

From the basic theory of elliptic partial differential equations, the solution of (4.23) exists and the following estimate holds

$$\begin{aligned} \|\nabla\left(p^2_n - p^1_n\right)\|_2^2 \leqslant C\big(&\|P_n\left(v^2_n\cdot\nabla v^2_n\right) - P_n\left(v^1_n\cdot\nabla v^1_n\right)\|_2^2 \\ &+ \|\operatorname{div}\left(P_n\left(\overline{\mu^2_n} Dv^2_n\right)\right) - \operatorname{div}\left(P_n\left(\overline{\mu^1_n} Dv^1_n\right)\right)\|_2^2\big). \end{aligned} \tag{4.27}$$

Let us analyse terms of the right-hand side of (4.27) separately. First by using the triangle inequality, Hölder inequality and (4.14) we get

$$\begin{aligned}\left\|P_n\left(v_n^2\cdot\nabla v_n^2\right)-P_n\left(v_n^1\cdot\nabla v_n^1\right)\right\|_2 &\leqslant \left\|(v_n^2-v_n^1)\cdot\nabla v_n^2 - v_n^1\cdot\nabla(v_n^1-v_n^2)\right\|_2\\ \leqslant \left\|\nabla v_n^2\right\|_\infty\left\|v_n^2-v_n^1\right\|_2+\left\|v_n^1\right\|_\infty\left\|\nabla(v_n^1-v_n^2)\right\|_2 &\leqslant C(\left\|v_n^2\right\|_2+\left\|v_n^1\right\|_2)\left\|v_n^2-v_n^1\right\|_2.\end{aligned} \tag{4.28}$$

Now, we go back to analysing the second term of the right-hand side of (4.27). Again, by using (4.14) we get

$$\left\|\operatorname{div}\left(P_n\left(\overline{\mu_n^2}Dv_n^2\right)\right)-\operatorname{div}\left(P_n\left(\overline{\mu_n^1}Dv_n^1\right)\right)\right\|_2\leqslant C\left\|\overline{\mu_n^2}Dv_n^2-\overline{\mu_n^1}Dv_n^1\right\|_2.$$

By using the triangle inequality and the Hölder inequality we obtain

$$\begin{aligned}&\left\|\operatorname{div}\left(P_n\left(\overline{\mu_n^2}Dv_n^2\right)\right)-\operatorname{div}\left(P_n\left(\overline{\mu_n^1}Dv_n^1\right)\right)\right\|_2\\ &\leqslant C\left\|(\overline{\mu_n^2}-\overline{\mu_n^1})D(v_n^2)-\overline{\mu_n^1}D(v_n^1-v_n^2)\right\|_2\\ &\leqslant C\big(\left\|D(v_n^2)\right\|_\infty\left\|\overline{\mu_n^2}-\overline{\mu_n^1}\right\|_2+\left\|\overline{\mu_n^1}\right\|_\infty\left\|D(v_n^1-v_n^2)\right\|_2\big).\end{aligned}$$

With the help of (4.14) we have

$$\begin{aligned}&\left\|\operatorname{div}\left(P_n\left(\overline{\mu_n^2}Dv_n^2\right)\right)-\operatorname{div}\left(P_n\left(\overline{\mu_n^1}Dv_n^1\right)\right)\right\|_2\\ &\qquad\leqslant C\left(\left\|v_n^2\right\|_2\left\|\overline{\mu_n^2}-\overline{\mu_n^1}\right\|_2+\left\|\overline{\mu_n^1}\right\|_\infty\left\|v_n^2-v_n^1\right\|_2\right).\end{aligned} \tag{4.29}$$

We see that (4.26), (4.9), (4.11) and (4.14) imply

$$\left\|\overline{\mu_n^1}\right\|_\infty=\left\|\frac{\Psi_t\left(\operatorname{Re}(b_n^1)\right)}{\Phi_t\left(\operatorname{Re}(\omega_n^1)\right)}\right\|_\infty\leqslant\frac{\frac12 b_{\min}^t+\|b_n^1\|_\infty}{\frac12\omega_{\min}^t}\leqslant\frac{b_{\min}^t+2C\left\|b_n^1\right\|_2}{\omega_{\min}^t} \tag{4.30}$$

and

$$\begin{aligned}\left\|\overline{\mu_n^2}-\overline{\mu_n^1}\right\|_2&=\left\|\frac{\Psi_t(\operatorname{Re}b_n^2)-\Psi_t(\operatorname{Re}b_n^1)}{\Phi_t(\operatorname{Re}\omega_n^2)}-\Psi_t(\operatorname{Re}b_n^1)\frac{\Phi_t(\operatorname{Re}\omega_n^2)-\Phi_t(\operatorname{Re}\omega_n^1)}{\Phi_t(\operatorname{Re}\omega_n^1)\Phi_t(\operatorname{Re}\omega_n^2)}\right\|_2\\ \leqslant\frac{2}{\omega_{\min}^t}\left\|\Psi_t(\operatorname{Re}b_n^2)-\Psi_t(\operatorname{Re}b_n^1)\right\|_2&+\frac{\frac12 b_{\min}^t+C\left\|b_n^1\right\|_2}{\left(\frac12\omega_{\min}^t\right)^2}\left\|\Phi_t(\operatorname{Re}\omega_n^2)-\Phi_t(\operatorname{Re}\omega_n^1)\right\|_2.\end{aligned}$$

Now we use the fact that functions Ψ_t, Φ_t are lipschitz continuous and obtain

$$\left\|\overline{\mu_n^2} - \overline{\mu_n^1}\right\|_2 \leqslant C \left(\frac{2}{\omega_{\min}^t} \left\|b_n^2 - b_n^1\right\|_2 + \frac{\frac{1}{2} b_{\min}^t + C \left\|b_n^1\right\|_2}{\left(\frac{1}{2}\omega_{\min}^t\right)^2} \left\|\omega_n^2 - \omega_n^1\right\|_2 \right). \tag{4.31}$$

Thus by plugging (4.31) and (4.30) into (4.29) we get

$$\begin{aligned} &\left\|\operatorname{div}\left(P_n\left(\overline{\mu_n^2} D(P_n v_n^2)\right)\right) - \operatorname{div}\left(P_n\left(\overline{\mu_n^1} D(P_n v_n^1)\right)\right)\right\|_2 \\ &\quad \leqslant C_1 \Big(\left\|v_n^2 - v_n^1\right\|_2 + \left\|b_n^2 - b_n^1\right\|_2 + \left\|\omega_n^2 - \omega_n^1\right\|_2 \Big), \end{aligned} \tag{4.32}$$

where $C_1 = C_1\big(\left\|b_n^1\right\|_2, \left\|v_n^2\right\|_2, b_{\min}^t, \omega_{\min}^t\big)$. Therefore by using (4.32), (4.28), (4.27) with the help of (4.25), (4.17) we get

$$\left\|\nabla\left(p_n^2 - p_n^1\right)\right\|_2^2 \leqslant C_2 \Big(\sum_{j=1}^d \sum_{|k|<n} |\alpha_{k,j}^{n,2} - \alpha_{k,j}^{n,1}|^2 + |\beta_k^{n,2} - \beta_{k,j}^{n,1}|^2 + \sum_{|k|<n} |\gamma_k^{n,2} - \gamma_k^{n,1}|^2 \Big),$$

where

$$C_2 = C_2 \left(\sum_{|k|<n} |\gamma_k^{n,1}|^2, \sum_{j=1}^d \sum_{|k|<n} |\alpha_{k,j}^{n,1}|^2, \sum_{j=1}^d \sum_{|k|<n} |\alpha_{k,j}^{n,2}|^2, b_{\min}^t, \omega_{\min}^t \right).$$

The verification of the Lipschitz condition for other terms is analogous to the conducted calculations. We will give one more estimate

$$\begin{aligned} &\left|\left(P_n\left(\overline{\mu_n^2} |Dv_n^2|^2\right), e^{-2\pi i k \cdot}\right) - \left(P_n\left(\overline{\mu_n^1} |Dv_n^1|^2\right), e^{-2\pi i k \cdot}\right)\right| \\ &\qquad\qquad \leqslant C \left\|P_n\left(\overline{\mu_n^2} |Dv_n^2|^2\right) - P_n\left(\overline{\mu_n^1} |Dv_n^1|^2\right)\right\|_2. \end{aligned}$$

By Hölder's inequality, we get

$$\begin{aligned} &\left\|P_n\left(\overline{\mu_n^2} |Dv_n^2|^2\right) - P_n\left(\overline{\mu_n^1} |Dv_n^1|^2\right)\right\|_2 \\ &\quad \leqslant \left\|\overline{\mu_n^2} - \overline{\mu_n^1}\right\|_2 \left\|Dv_n^2\right\|_\infty^2 + \left\|\overline{\mu_n^1}\right\|_\infty \left\|Dv_n^2 + Dv_n^1\right\|_\infty \left\|Dv_n^2 - Dv_n^1\right\|_2. \end{aligned}$$

Now, based on (4.30), (4.31), (4.14) and (4.25) we get

$$\begin{aligned} &\left\|P_n\left(\overline{\mu_n^2} |DP_n v_n^2|^2\right) - P_n\left(\overline{\mu_n^1} |DP_n v_n^1|^2\right)\right\|_2 \\ &\qquad\qquad \leqslant \widetilde{C} \Big(\left\|v_n^2 - v_n^1\right\|_2 + \left\|b_n^2 - b_n^1\right\|_2 + \left\|\omega_n^2 - \omega_n^1\right\|_2 \Big), \end{aligned}$$

where

$$\widetilde{C} = \widetilde{C}\left(\sum_{|k|<n} |\gamma_k^{n,1}|^2, \sum_{j=1}^{d}\sum_{|k|<n} |\alpha_{k,j}^{n,1}|^2, \sum_{j=1}^{d}\sum_{|k|<n} |\alpha_{k,j}^{n,2}|^2, b_{\min}^t, \omega_{\min}^t\right).$$

Because all right-hand side's terms in (4.19) - (4.21) are locally lipschitz continuous, the existence of a unique solution for some $T_n > 0$ follows from the Cauchy-Lipschitz theorem. Now let us multiply equations (4.19)-(4.21) by $e^{2\pi ikx}$ and make the summation over $|k| < n$:

$$\partial_t v_n + P_n\left(v_n \cdot \nabla v_n\right) - \operatorname{div}\left(P_n\left(\overline{\mu_n} D v_n\right)\right) + \nabla p_n = 0, \tag{4.33}$$

$$\partial_t \omega_n + P_n\left(v_n \cdot \nabla \omega_n\right) - \operatorname{div}\left(P_n\left(\overline{\mu_n} \nabla \omega_n\right)\right) = -\kappa_2 P_n\left(\omega_n^2\right), \tag{4.34}$$

$$\partial_t b_n + P_n\left(v_n \cdot \nabla b_n\right) - \operatorname{div}\left(P_n\left(\overline{\mu_n} \nabla b_n\right)\right) = -P_n\left(b_n \omega_n\right) + P_n\left(\overline{\mu_n} |D v_n|^2\right), \tag{4.35}$$

$$v_n(0,x) = P_n v_0(x), \quad \omega_n(0,x) = P_n \omega_0(x), \quad b_n(0,x) = P_n b_0(x).$$

Now, by applying the divergence operator to equation (4.33) and by using (4.23), (4.15) we get

$$\operatorname{div} v_n = 0. \tag{4.36}$$

Moreover by taking imaginary parts of the system (4.33) - (4.35) and having in mind that initial data is real-valued, it is easy to check that solutions (v_n, ω_n, b_n) are also real-valued. We will provide more details for v_n. Let us apply Im to equation (4.33), multiply result by $\operatorname{Im} v_n$ and integrate over $\mathbb{T}^d$ to obtain

$$\begin{aligned}(\partial_t \operatorname{Im} v_n, \operatorname{Im} v_n) &+ \left(P_n\left(\operatorname{Im} v_n \cdot \nabla \operatorname{Re} v_n\right), \operatorname{Im} v_n\right) + \left(P_n\left(\operatorname{Re} v_n \cdot \nabla \operatorname{Im} v_n\right), \operatorname{Im} v_n\right) \\ &+ \left(P_n\left(\overline{\mu_n} D \operatorname{Im} v_n\right), D \operatorname{Im} v_n\right) - \left(\operatorname{Im} p_n, \operatorname{div}(\operatorname{Im} v_n)\right) = 0.\end{aligned}$$

The last term on the left-hand side is zero due to (4.36). By (4.18), (4.9), (4.11) we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\operatorname{Im} v_n\|_2^2 + \mu_{\min}^t \|D \operatorname{Im} v_n\|_2^2 &\leqslant \|\nabla \operatorname{Re} v_n\|_\infty \|\operatorname{Im} v_n\|_2^2 \\ &+ \|\operatorname{Re} v_n\|_\infty \|\operatorname{Im} v_n\|_2 \|\nabla \operatorname{Im} v_n\|_2 .\end{aligned}$$

By using the Young inequality we get

$$\frac{1}{2}\frac{d}{dt}\|\operatorname{Im} v_n\|_2^2 + \frac{\mu^t_{\min}}{2}\|\nabla \operatorname{Im} v_n\|_2^2 \leqslant \|\nabla \operatorname{Re} v_n\|_\infty \|\operatorname{Im} v_n\|_2^2 + \frac{1}{2\mu^t_{\min}}\|\operatorname{Re} v_n\|_\infty^2 \|\operatorname{Im} v_n\|_2^2 + \frac{\mu^t_{\min}}{2}\|\nabla \operatorname{Im} v_n\|_2^2 .$$

Using the Grönwall lemma and the fact that initial data is real-valued we conclude that

$$\|\operatorname{Im} v_n(t)\|_2^2 = 0 \quad \forall t \in [0, T_n)$$

and thus that velocity is real-valued. A similar approach can be applied to ω_n and b_n. Thus, system (4.33) - (4.35) can be rewritten in a following way:

$$\operatorname{div} v_n = 0, \tag{4.37}$$

$$\partial_t v_n + P_n\left(v_n \cdot \nabla v_n\right) - \operatorname{div}\left(P_n\left(\mu_n D v_n\right)\right) + \nabla p_n = 0, \tag{4.38}$$

$$\partial_t \omega_n + P_n\left(v_n \cdot \nabla \omega_n\right) - \operatorname{div}\left(P_n\left(\mu_n \nabla \omega_n\right)\right) = -\kappa_2 P_n\left(\omega_n^2\right), \tag{4.39}$$

$$\partial_t b_n + P_n\left(v_n \cdot \nabla b_n\right) - \operatorname{div}\left(P_n\left(\mu_n \nabla b_n\right)\right) = -P_n\left(b_n \omega_n\right) + P_n\left(\mu_n |D v_n|^2\right), \tag{4.40}$$

$$v_n(0,x) = P_n v_0(x), \quad \omega_n(0,x) = P_n \omega_0(x), \quad b_n(0,x) = P_n b_0(x), \tag{4.41}$$

where:

$$\mu_n = \frac{\Psi_t(b_n)}{\Phi_t(\omega_n)}. \tag{4.42}$$

From the equation (4.16) it is clear that functions (v_n, ω_n, b_n) are smooth with respect to the spatial coordinates. By employing a standard iterative approach from the theory of ODE's (C^k right-hand side implies C^{k+1} solution) applied to the system (4.19)-(4.21), it is easy to conclude that (v_n, ω_n, b_n) are also smooth with respect to time.

4.2.3. Energy estimates

Before we start deriving energy estimates we need to establish some relations between J^s and the derivative. Let $f : \mathbb{T}^d \to \mathbb{C}$, $w : \mathbb{T}^d \to \mathbb{C}^d$ such that $P_n f = f$ and $P_n w = w$.

Let us recall that $\hat{f}(k) = \int_{\mathbb{T}^d} f(x)e^{-2\pi i x\cdot k}dx$. Then we have

$$\begin{aligned}\frac{\partial J^s f}{\partial x_i}(x) &= \frac{\partial}{\partial x_i} \sum_{k\in\mathbb{Z}^d:|k|<n} \left(1+4\pi^2|k|^2\right)^{s/2} e^{2\pi i x\cdot k}\hat{f}(k)\\ &= \sum_{k\in\mathbb{Z}^d:|k|<n} \left(1+4\pi^2|k|^2\right)^{s/2} e^{2\pi i x\cdot k}(2\pi i k_i)\hat{f}(k).\end{aligned}$$

Now using the properties of the Fourier transform acting on a derivative we get

$$\frac{\partial J^s f}{\partial x_i}(x) = \sum_{k\in\mathbb{Z}^d:|k|<n} \left(1+4\pi^2|k|^2\right)^{s/2} e^{2\pi i x\cdot k}\widehat{\left(\frac{\partial f}{\partial x_i}\right)}(k) = J^s\left(\frac{\partial f}{\partial x_i}\right)(x).$$

Thus we have

$$\nabla J^s f = J^s\nabla f, \qquad DJ^s f = J^s Df, \qquad J^s \operatorname{div} w = \operatorname{div} J^s w, \qquad \Delta J^s f = J^s \Delta f. \tag{4.43}$$

Furthermore, it can similarly be shown that

$$P_n J^s f = J^s P_n f. \tag{4.44}$$

In the next part, we will be calculating various inner products in $L^2(\mathbb{T}^d)$. Thus to simplify reasoning it is beneficial to observe that if the function f is a real-valued function then $J^s f$ is also real-valued. Before we proceed with energy estimates we need to introduce notation regarding constants dependent on time. In later parts of the proof positive constants $C(\eta_1, ..., \eta_m, t)$ (dependent on the time) are such that

$$C(\eta_1, ..., \eta_m, t) = \widetilde{C}(\eta_1, ..., \eta_m)(1+t)^\gamma \tag{4.45}$$

for some $\gamma \in \mathbb{R}$. Let us apply the J^s operator to equation (4.38), multiply the result by $J^s v_n$, and integrate over $\mathbb{T}^d$. From this we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|J^s v_n\|_2^2 + \left(J^s P_n\left(v_n\cdot\nabla v_n\right), J^s v_n\right) - \left(J^s \operatorname{div} P_n\left(\mu_n D v_n\right), J^s v_n\right)&\\ = -\left(J^s\nabla p_n, J^s v_n\right).&\end{aligned}$$

Using properties (4.43), (4.44), integration by parts, and the fact that $\operatorname{div} v_n = 0$ we get

$$\frac{1}{2}\frac{d}{dt}\|v_n\|^2_{H^s} - \left(J^s \operatorname{div}\left(\mu_n D v_n\right), J^s v_n\right) = -\left(J^s\left(v_n \cdot \nabla v_n\right), J^s v_n\right). \tag{4.46}$$

Now using the Hölder inequality and Lemma 1.3.2 implies

$$\left|\left(J^s\left(v_n \cdot \nabla v_n\right), J^s v_n\right)\right| \leqslant C\|v_n\|^2_{H^s}\|\nabla v_n\|_{H^s}. \tag{4.47}$$

Using properties (4.43) and integration by parts yields

$$-\left(J^s \operatorname{div}\left(\mu_n D v_n\right), J^s v_n\right) = \left(J^s\left(\mu_n D v_n\right), \nabla\left(J^s v_n\right)\right).$$

Now, we rewrite the expression in a way that will enable us to use the commutator estimate:

$$\left(J^s\left(\mu_n D v_n\right), \nabla J^s v_n\right) = \left(\mu_n D J^s v_n, \nabla J^s v_n\right) + \left(\left[J^s, \mu_n\right] D v_n, \nabla J^s v_n\right), \tag{4.48}$$

where $\left[J^s, \mu_n\right] D v_n := J^s(\mu_n D v_n) - \mu_n J^s D v_n$. We want to estimate the above expression from below. Using the Hölder inequality, Lemmas 1.3.3, 1.3.7, we get

$$\begin{aligned}\left(\left[J^s, \mu_n\right] D v_n, \nabla J^s v_n\right) &\leqslant C\left(\|\nabla \mu_n\|_\infty \|\nabla v_n\|_{H^{s-1}} + \|\mu_n\|_{H^s}\|\nabla v_n\|_\infty\right)\|\nabla v_n\|_{H^s} \\ &\leqslant C\left(\|\nabla \mu_n\|_\infty \|v_n\|_{H^s} + \|\mu_n\|_{H^s}\|\nabla v_n\|_\infty\right)\|\nabla v_n\|_{H^s}.\end{aligned} \tag{4.49}$$

For now, we will leave the inequality in the above form. By definitions (4.3), (4.9), (4.11) and (4.42) we have

$$\left(\mu_n D J^s v_n, \nabla J^s v_n\right) = \left(\mu_n D J^s v_n, D J^s v_n\right) \geqslant \mu^t_{\min}\|D v_n\|^2_{H^s}. \tag{4.50}$$

Let us rewrite the right-hand side in the following way

$$\begin{aligned}\|D v_n\|^2_{H^s} &= \sum_{i,j=1}^{d}\left(\frac{(J^s v_n)_{i,j} + (J^s v_n)_{j,i}}{2}, \frac{(J^s v_n)_{i,j} + (J^s v_n)_{j,i}}{2}\right) \\ &= \frac{1}{2}\sum_{i,j=1}^{d}\left((J^s v_n)_{i,j}, (J^s v_n)_{i,j}\right) + \left((J^s v_n)_{i,j}, (J^s v_n)_{j,i}\right).\end{aligned}$$

By performing integration by parts and using the fact that $\operatorname{div} v_n = 0$ in combination with (4.43) and (4.50) we get

$$(\mu_n D J^s v_n, \nabla J^s v_n) \geqslant \frac{1}{2}\mu^t_{\min} \|\nabla v_n\|^2_{H^s}.$$

By combining (4.48), (4.49) and the above inequality, we obtain

$$\begin{aligned}(\mu_n D v_n, \nabla v_n)_{H^s} \geqslant \frac{\mu^t_{\min}}{2} \|\nabla v_n\|^2_{H^s} - C\|\nabla \mu_n\|_\infty \|v_n\|_{H^s} \|\nabla v_n\|_{H^s} \\ - C\|\mu_n\|_{H^s} \|\nabla v_n\|_\infty \|\nabla v_n\|_{H^s}.\end{aligned} \tag{4.51}$$

Using estimates (4.47) and (4.51) in (4.46) implies

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|v_n\|^2_{H^s} + \frac{\mu^t_{\min}}{2}\|\nabla v_n\|^2_{H^s} \\ &\quad\leqslant C\Big(\|\nabla \mu_n\|_\infty \|v_n\|_{H^s} \|\nabla v_n\|_{H^s} + \|\mu_n\|_{H^s} \|\nabla v_n\|_\infty \|\nabla v_n\|_{H^s} + \|v_n\|^2_{H^s} \|\nabla v_n\|_{H^s} \Big).\end{aligned}$$

Now, according to Lemma 1.3.7 we can express $\|\nabla v_n\|^2_{H^s}$ in the following way

$$\|\nabla v_n\|^2_{H^s} = \|v_n\|^2_{H^{s+1}} - \|v_n\|^2_{H^s}.$$

We can rewrite inequality in the following way:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|v_n\|^2_{H^s} + \frac{\mu^t_{\min}}{2}\|v_n\|^2_{H^{s+1}} \leqslant \frac{\mu^t_{\min}}{2}\|v_n\|^2_{H^s} \\ &\quad + C\Big(\|\nabla \mu_n\|_\infty \|v_n\|_{H^s} \|v_n\|_{H^{s+1}} + \|\mu_n\|_{H^s} \|\nabla v_n\|_\infty \|v_n\|_{H^{s+1}} + \|v_n\|^2_{H^s} \|v_n\|_{H^{s+1}} \Big).\end{aligned} \tag{4.52}$$

Now, we will proceed with acquiring the estimate on ω_n. Let us apply J^s operator to the equation (4.39) and take the inner product with with $J^s\omega_n$:

$$\frac{1}{2}\frac{d}{dt}\|\omega_n\|^2_{H^s} + (\mu_n \nabla \omega_n, \nabla \omega_n)_{H^s} = -(v_n \cdot \nabla \omega_n, \omega_n)_{H^s} - \kappa_2 \left(\omega_n^2, \omega_n\right)_{H^s}. \tag{4.53}$$

Proceeding as in (4.51), we obtain

$$(\mu_n \nabla \omega_n, \nabla \omega_n)_{H^s} = (\mu_n J^s \nabla \omega_n, J^s \nabla \omega_n) + (J^s(\mu_n \nabla \omega_n) - \mu_n J^s(\nabla \omega_n), J^s \nabla \omega_n).$$

Thus using the Hölder inequality and Lemma 1.3.3 we get

$$\begin{aligned}(\mu_n \nabla\omega_n, \nabla\omega_n)_{H^s} \geqslant \mu^t_{\min} \|\nabla\omega_n\|^2_{H^s} - C\|\nabla\mu_n\|_\infty \|\omega_n\|_{H^s} \|\nabla\omega_n\|_{H^s} \\ - C\|\mu_n\|_{H^s} \|\nabla\omega_n\|_\infty \|\nabla\omega_n\|_{H^s}.\end{aligned} \tag{4.54}$$

Now, using the Hölder inequality in combination with Lemma 1.3.2, we treat remaining nonlinearities in a following way:

$$|(v_n \cdot \nabla\omega_n, \omega_n)_{H^s}| \leqslant \|v_n\|_{H^s} \|\omega_n\|_{H^s} \|\nabla\omega_n\|_{H^s}, \tag{4.55}$$

$$\left|\left(\omega_n^2, \omega_n\right)_{H^s}\right| \leqslant C\|\omega_n\|^3_{H^s}. \tag{4.56}$$

Applying inequalities (4.54), (4.55), and (4.56) to (4.53) gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\omega_n\|^2_{H^s} + \mu^t_{\min}\|\nabla\omega_n\|^2_{H^s} \leqslant C\Big(&\|\nabla\mu_n\|_\infty \|\omega_n\|_{H^s} \|\nabla\omega_n\|_{H^s} \\ &+ \|\mu_n\|_{H^s} \|\nabla\omega_n\|_\infty \|\nabla\omega_n\|_{H^s} + \|\omega_n\|^3_{H^s} + \|v_n\|_{H^s} \|\omega_n\|_{H^s} \|\nabla\omega_n\|_{H^s}\Big).\end{aligned}$$

Using Lemma 1.3.7, we can rewrite it in a more suitable form

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\omega_n\|^2_{H^s} + \mu^t_{\min}\|\omega_n\|^2_{H^{s+1}} \leqslant \mu^t_{\min}\|\omega_n\|^2_{H^s} + C\Big(&\|\nabla\mu_n\|_\infty \|\omega_n\|_{H^s} \|\omega_n\|_{H^{s+1}} \\ &+ \|\mu_n\|_{H^s} \|\nabla\omega_n\|_\infty \|\omega_n\|_{H^{s+1}} + \|\omega_n\|^3_{H^s} + \|v_n\|_{H^s} \|\omega_n\|_{H^s} \|\omega_n\|_{H^{s+1}}\Big).\end{aligned} \tag{4.57}$$

Now estimates for b_n will be provided. As before, let us apply J^s operator to the equation (4.40) and take the inner product with with $J^s b_n$:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|b_n\|^2_{H^s} + (\mu_n\nabla b_n, \nabla b_n)_{H^s} = &-(v_n \cdot \nabla b_n, b_n)_{H^s} - (b_n\omega_n, b_n)_{H^s} \\ &+ \left(\mu_n|D(v_n)|^2, b_n\right)_{H^s}.\end{aligned} \tag{4.58}$$

Proceeding as before, with the use of Lemmas 1.3.3 and 1.3.2 we get:

$$|(v_n \cdot \nabla b_n, b_n)_{H^s}| \leqslant \|v_n\|_{H^s} \|b_n\|_{H^s} \|\nabla b_n\|_{H^s}, \tag{4.59}$$

$$|(b_n\omega_n, b_n)_{H^s}| \leqslant C\|b_n\|^2_{H^s} \|\omega_n\|_{H^s}, \tag{4.60}$$

$$\begin{aligned}(\mu_n\nabla b_n, \nabla b_n)_{H^s} \geqslant \mu^t_{\min}\|\nabla b_n\|^2_{H^s} - C\|\nabla\mu_n\|_\infty \|b_n\|_{H^s} \|\nabla b_n\|_{H^s} \\ - C\|\mu_n\|_{H^s} \|\nabla b_n\|_\infty \|\nabla b_n\|_{H^s}.\end{aligned} \tag{4.61}$$

Now we provide the estimate for the last term of r.h.s. of (4.58). Using Lemmas 1.3.2 and 1.3.1 yields

$$\begin{aligned}\left|\left(\mu_n |D(v_n)|^2, b_n\right)_{H^s}\right| &\leqslant \|\mu_n\|_{H^s} \left\||D(v_n)|^2\right\|_{H^s} \|b_n\|_{H^s} \\ &\leqslant C \|\mu_n\|_{H^s} \|\nabla v_n\|_\infty \|\nabla v_n\|_{H^s} \|b_n\|_{H^s}.\end{aligned} \tag{4.62}$$

Finally, by using estimates (4.59), (4.60), (4.61), (4.62) in (4.58) and applying Lemma 1.3.7 we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|b_n\|_{H^s}^2 + \mu_{\min}^t \|b_n\|_{H^{s+1}}^2 \leqslant \mu_{\min}^t \|b_n\|_{H^s}^2 + C\Bigg(\|\nabla \mu_n\|_\infty \|b_n\|_{H^s} \|b_n\|_{H^{s+1}} \\ &\quad + \|\mu_n\|_{H^s} \|\nabla b_n\|_\infty \|b_n\|_{H^{s+1}} + \|b_n\|_{H^s}^2 \|\omega_n\|_{H^s} + \|v_n\|_{H^s} \|b_n\|_{H^s} \|b_n\|_{H^{s+1}} \\ &\quad + \|\mu_n\|_{H^s} \|\nabla v_n\|_\infty \|v_n\|_{H^{s+1}} \|b_n\|_{H^s} \Bigg).\end{aligned} \tag{4.63}$$

By summing inequalities (4.52), (4.57) and (4.63) we get:

$$\begin{aligned}&\frac{d}{dt}\|v_n,\omega_n,b_n\|_{H^s}^2 + \mu_{\min}^t \|v_n,\omega_n,b_n\|_{H^{s+1}}^2 \leqslant \mu_{\min}^t \|v_n,\omega_n,b_n\|_{H^s}^2 \\ &+ C\Bigg(\|\mu_n\|_{H^s} \left(\|\nabla v_n\|_\infty + \|\nabla \omega_n\|_\infty + \|\nabla b_n\|_\infty\right) \|v_n,\omega_n,b_n\|_{H^{s+1}} + \|v_n,\omega_n,b_n\|_{H^s}^3 \\ &+ \|\nabla \mu_n\|_\infty \|v_n,\omega_n,b_n\|_{H^s} \|v_n,\omega_n,b_n\|_{H^{s+1}} + \|v_n,\omega_n,b_n\|_{H^s}^2 \|v_n,\omega_n,b_n\|_{H^{s+1}} \\ &+ \|\mu_n\|_{H^s} \|\nabla v_n\|_\infty \|v_n\|_{H^{s+1}} \|b_n\|_{H^s} \Bigg),\end{aligned} \tag{4.64}$$

where $\|v_n,\omega_n,b_n\|_{H^s}^2 = \|v_n\|_{H^s}^2 + \|\omega_n\|_{H^s}^2 + \|b_n\|_{H^s}^2$. Now let us see that by definition (4.42) and (4.9)-(4.12) we have

$$\begin{aligned}\|\nabla \mu_n\|_\infty = \left\|\nabla \frac{\Psi_t(b_n)}{\Phi_t(\omega_n)}\right\|_\infty &= \left\|\frac{\Psi_t'(b_n)}{\Phi_t(\omega_n)}\nabla b_n - \frac{\Psi_t(b_n)}{\Phi_t^2(\omega_n)}\Phi_t'(\omega_n)\nabla\omega_n\right\|_\infty \\ &\leqslant C(\omega_{\min},t)\left(\|\nabla b_n\|_\infty + \|\Psi_t(b_n)\|_\infty \|\nabla\omega_n\|_\infty\right) \\ &\leqslant C(\omega_{\min},t)\left(\|\nabla b_n\|_\infty + (b_{\min} + \|b_n\|_\infty)\|\nabla\omega_n\|_\infty\right),\end{aligned}$$

where constant $C(\omega_{\min},t)$ is as in (4.45). Lemma 1.3.4 yields

$$\|\nabla \mu_n\|_\infty \leqslant C(\omega_{\min},t)\left(\|\nabla b_n\|_\infty + (b_{\min} + \|b_n\|_{H^s})\|\nabla\omega_n\|_\infty\right).$$

To simplify the expressions we will introduce the polynomial notation: for $k \in \mathbb{R}_+$ we define $P_k(t)$ in the following way:

$$P_k(t) = \left(1 + \|v_n(t)\|_{H^s}^2 + \|\omega_n(t)\|_{H^s}^2 + \|b_n(t)\|_{H^s}^2\right)^{k/2}. \tag{4.65}$$

Thus we can write

$$\|\nabla \mu_n\|_\infty \leqslant C(\omega_{\min}, b_{\min}, t) P_1(t) \left(\|\nabla b_n\|_\infty + \|\nabla \omega_n\|_\infty\right). \tag{4.66}$$

Now using (4.66) and (4.65) in (4.64) we get

$$\begin{aligned} &\frac{d}{dt} \|v_n, \omega_n, b_n\|_{H^s}^2 + \mu_{\min}^t \|v_n, \omega_n, b_n\|_{H^{s+1}}^2 \leqslant \mu_{\min}^t P_2(t) \\ &\quad + C\Bigg(\|\mu_n\|_{H^s} \left(\|\nabla v_n\|_\infty + \|\nabla \omega_n\|_\infty + \|\nabla b_n\|_\infty\right) \|v_n, \omega_n, b_n\|_{H^{s+1}} + P_3(t) \\ &\quad + P_2(t) \left(\|\nabla b_n\|_\infty + \|\nabla \omega_n\|_\infty\right) \|v_n, \omega_n, b_n\|_{H^{s+1}} + P_2(t) \|v_n, \omega_n, b_n\|_{H^{s+1}} \\ &\quad + \|\mu_n\|_{H^s} \|\nabla v_n\|_\infty P_1(t) \|v_n\|_{H^{s+1}} \Bigg). \end{aligned} \tag{4.67}$$

By using the properties of $P_k(t)$ we can write

$$\begin{aligned} &\frac{d}{dt} \|v_n, \omega_n, b_n\|_{H^s}^2 + \mu_{\min}^t \|v_n, \omega_n, b_n\|_{H^{s+1}}^2 \leqslant \mu_{\min}^t P_2(t) + C\Bigg(P_2(t) \|v_n, \omega_n, b_n\|_{H^{s+1}} \\ &+ P_3(t) + \left(P_2(t) + \|\mu_n\|_{H^s} P_1(t)\right) \left(\|\nabla v_n\|_\infty + \|\nabla \omega_n\|_\infty + \|\nabla b_n\|_\infty\right) \|v_n, \omega_n, b_n\|_{H^{s+1}} \Bigg). \end{aligned} \tag{4.68}$$

Now, we will continue the proof assuming that $s \in \left(\frac{d}{2}, \frac{d}{2} + 1\right]$ - because in other cases we have $\|\nabla f\|_\infty \leqslant C \|f\|_{H^s}$ and subsequent estimates simplify. Thus by Lemma 1.3.11 we get:

$$\begin{aligned} &\frac{d}{dt} \|v_n, \omega_n, b_n\|_{H^s}^2 + \mu_{\min}^t \|v_n, \omega_n, b_n\|_{H^{s+1}}^2 \leqslant \mu_{\min}^t P_2(t) + C\Big(P_2(t) \|v_n, \omega_n, b_n\|_{H^{s+1}} \\ &+ P_3(t) + \left(P_2(t) + \|\mu_n\|_{H^s} P_1(t)\right) P_{\frac{1}{2}\left(s - \frac{d}{2}\right)}(t) \|v_n, \omega_n, b_n\|_{H^{s+1}}^{1 - \frac{1}{2}\left(s - \frac{d}{2}\right)} \|v_n, \omega_n, b_n\|_{H^{s+1}} \Big). \end{aligned} \tag{4.69}$$

In the above inequality term $\|\mu_n\|_{H^s}$ remains not estimated. First, let us consider three auxiliary estimates that follow from Lemma 1.3.5, Lemma 1.3.4, (4.10) and (4.9)

$$\begin{aligned} \|\Psi_t(b_n)\|_{H^s} &\leqslant \left\|\Psi_t(b_n) - \frac{1}{2}b^t_{\min}\right\|_{H^s} + \left\|\frac{1}{2}b^t_{\min}\right\|_{H^s} \\ &\leqslant C\left\|\Psi'_t\right\|_{C^{\lceil s\rceil}}(1+\|b_n\|_{H^s})^{\lceil s\rceil}\|b_n\|_{H^s} + \left\|\frac{1}{2}b^t_{\min}\right\|_{H^s} \\ &\leqslant C\left(1+\|b_n\|_{H^s}\right)^{\lceil s\rceil}\|b_n\|_{H^s} + C, \end{aligned} \tag{4.70}$$

where $C = C(s, b_{\min}, t)$ is a rational function dependent on time t, finite $\forall t \geqslant 0$ (see (4.45)). Similarly from Lemma 1.3.5, Lemma 1.3.4, (4.12) and (4.11) it follows

$$\begin{aligned} \left\|\frac{1}{\Phi_t(\omega_n)}\right\|_{H^s} &\leqslant \left\|\frac{1}{\Phi_t(\omega_n)} - \frac{1}{\frac{1}{2}\omega^t_{\min}}\right\|_{H^s} + \left\|\frac{1}{\frac{1}{2}\omega^t_{\min}}\right\|_{H^s} \\ &\leqslant C\left\|\left(\frac{1}{\Phi_t}\right)'\right\|_{C^{\lceil s\rceil}}(1+\|\omega_n\|_{H^s})^{\lceil s\rceil}\|\omega_n\|_{H^s} + \left\|\frac{2}{\omega^t_{\min}}\right\|_{H^s} \\ &\leqslant C\left(1+\|\omega_n\|_{H^s}\right)^{\lceil s\rceil}\|\omega_n\|_{H^s} + C, \end{aligned} \tag{4.71}$$

where $C = C(s, \omega_{\min}, \omega_{\max}, t)$ is as in (4.45). Now using the obtained estimates in combination with Lemmas 1.3.1, 1.3.4 and definitions (4.11), (4.42) we can proceed with estimates on $\|\mu_n\|_{H^s}$ in the following way:

$$\begin{aligned} \|\mu_n\|_{H^s} = \left\|\frac{\Psi_t(b_n)}{\Phi_t(\omega_n)}\right\|_{H^s} &\leqslant C\left(\|\Psi_t(b_n)\|_{H^s}\left\|\frac{1}{\Phi_t(\omega_n)}\right\|_{\infty} + \|\Psi_t(b_n)\|_{\infty}\left\|\frac{1}{\Phi_t(\omega_n)}\right\|_{H^s}\right) \\ &\leqslant C\left(\frac{2}{\omega^t_{\min}}\|\Psi_t(b_n)\|_{H^s} + \|\Psi_t(b_n)\|_{H^s}\left\|\frac{1}{\Phi_t(\omega_n)}\right\|_{H^s}\right) \\ &\leqslant C_1\left((1+\|b_n\|_{H^s})^{\lceil s\rceil}\|b_n\|_{H^s} + 1\right)\cdot\left((1+\|\omega_n\|_{H^s})^{\lceil s\rceil}\|\omega_n\|_{H^s} + 1\right) \\ &\leqslant C_1 P_{2\lceil s\rceil+2}(t), \end{aligned} \tag{4.72}$$

where $C_1 = C_1(\omega_{\min}, \omega_{\max}, b_{\min}, t)$ is as in (4.45). Now let us use estimate (4.72) in (4.69):

$$\begin{aligned} \frac{d}{dt}\|v_n, \omega_n, b_n\|^2_{H^s} + \mu^t_{\min}\|v_n, \omega_n, b_n\|^2_{H^{s+1}} \leqslant \mu^t_{\min}P_2(t) + C\Big(P_2(t)\|v_n, \omega_n, b_n\|_{H^{s+1}} \\ + P_3(t) + \big(P_2(t) + P_{2\lceil s\rceil+3}(t)\big)P_{\frac{1}{2}\left(s-\frac{d}{2}\right)}(t)\|v_n, \omega_n, b_n\|^{2-\frac{1}{2}\left(s-\frac{d}{2}\right)}_{H^{s+1}}\Big). \end{aligned}$$

Using the fact that for $k_1 \geqslant k_2$ we have $P_{k_1}(t) \geqslant P_{k_2}(t)$ and Young's inequality we get

$$\begin{aligned}\frac{d}{dt}\|v_n,\omega_n,b_n\|_{H^s}^2 + \mu_{\min}^t \|v_n,\omega_n,b_n\|_{H^{s+1}}^2 &\leqslant C(b_{\min},\omega_{\min},\omega_{\max},t)\\ \cdot\Big(P_{2\lceil s\rceil+3+\frac{1}{2}\left(s-\frac{d}{2}\right)}(t)\|v_n,\omega_n,b_n\|_{H^{s+1}}^{2-\frac{1}{2}\left(s-\frac{d}{2}\right)} &+ P_3(t) + P_2(t)\|v_n,\omega_n,b_n\|_{H^{s+1}}\Big).\end{aligned} \tag{4.73}$$

Now, let us apply Young's inequality to the right-hand side with coefficients

$$\left(\frac{4}{s-\frac{d}{2}}, \frac{2}{2-\frac{1}{2}\left(s-\frac{d}{2}\right)}\right), \qquad (2,2)$$

to obtain

$$\begin{aligned}\frac{d}{dt}\|v_n,\omega_n,b_n\|_{H^s}^2 + \frac{\mu_{\min}^t}{2}\|v_n,\omega_n,b_n\|_{H^{s+1}}^2 &\leqslant C(b_{\min},\omega_{\min},\omega_{\max},t)\\ &\cdot\Big(P_{\left(2\lceil s\rceil+3+\frac{1}{2}\left(s-\frac{d}{2}\right)\right)\frac{4}{s-\frac{d}{2}}}(t) + P_4(t) + P_3(t)\Big).\end{aligned}$$

Now, let us introduce $\beta(s) > 1$ such that

$$2\beta(s) = \max\left\{4, \left(2\lceil s\rceil + 3 + \frac{1}{2}\left(s-\frac{d}{2}\right)\right)\frac{4}{s-\frac{d}{2}}\right\}.$$

Thus we have

$$\frac{d}{dt}\|v_n,\omega_n,b_n\|_{H^s}^2 + \frac{\mu_{\min}^t}{2}\|v_n,\omega_n,b_n\|_{H^{s+1}}^2 \leqslant C(b_{\min},\omega_{\min},\omega_{\max},s,t)P_{2\beta}(t). \tag{4.74}$$

Hence by definition (4.65) we get

$$\begin{aligned}\frac{d}{dt}\left(1+\|v_n,\omega_n,b_n\|_{H^s}^2\right) &+ \frac{\mu_{\min}^t}{2}\|v_n,\omega_n,b_n\|_{H^{s+1}}^2\\ &\leqslant C(b_{\min},\omega_{\min},\omega_{\max},s,t)\left(1+\|v_n,\omega_n,b_n\|_{H^s}^2\right)^{\beta}.\end{aligned} \tag{4.75}$$

By integrating from 0 to t the inequality follows

$$\begin{aligned}\frac{1}{-\beta+1}\left(1+\|v_n,\omega_n,b_n\|_{H^s}^2\right)^{-\beta+1} &\leqslant \frac{1}{-\beta+1}\left(1+\|P_nv_0,P_n\omega_0,P_nb_0\|_{H^s}^2\right)^{-\beta+1}\\ &+\int_0^t C(b_{\min},\omega_{\min},\omega_{\max},s,\tau)d\tau.\end{aligned}$$

After some transformations, we obtain the uniform estimate for approximated solutions

$$\|v_n,\omega_n,b_n\|_{H^s}^2 \leqslant \frac{1}{\left(\left(1+\|v_0,\omega_0,b_0\|_{H^s}^2\right)^{1-\beta}-(\beta-1)\int_0^t C(b_{\min},\omega_{\min},\omega_{\max},s,\tau)d\tau\right)^{\frac{1}{\beta-1}}}-1 \tag{4.76}$$

provided the denominator on the right-hand side is positive. Thus, let us define the existence time $T>0$ such that the following equality holds

$$\left(1-2^{-\beta+1}\right)\left(1+\|v_0,\omega_0,b_0\|_{H^s}^2\right)^{-\beta+1}=(\beta-1)\int_0^T C(b_{\min},\omega_{\min},\omega_{\max},s,\tau)d\tau. \tag{4.77}$$

By (4.77) in (4.76) we derive the estimate:

$$\|v_n,\omega_n,b_n\|_{H^s}^2 \leqslant 2\|v_0,\omega_0,b_0\|_{H^s}^2+1 \qquad \forall t\in[0,T]. \tag{4.78}$$

Additionally (4.75) and (4.78) imply that:

$$\int_0^T \|v_n,\omega,b_n\|_{H^{s+1}}^2\,d\tau \leqslant C\left(b_{\min},\omega_{\min},\omega_{\max},\|v_0,\omega_0,b_0\|_{H^s},s,T\right)<\infty. \tag{4.79}$$

To show the continuity of the solution, the estimate in the norm $L^2(0,T,H^{s-1}(\mathbb{T}^d))$ for the time derivative of the solution is required. We will derive the estimate only for b_n as the calculations for other variables are similar. Let us apply J^{s-1} to equation (4.40) and calculate the inner product with $\partial_t J^{s-1}b_n$

$$\begin{aligned}\|\partial_t b_n\|_{H^{s-1}}^2 =& -\left(v_n\nabla b_n,\partial_t b_n\right)_{H^{s-1}}+\left(\nabla\cdot\left(\mu_n\nabla b_n\right),\partial_t b_n\right)_{H^{s-1}}-\left(b_n\omega_n,\partial_t b_n\right)_{H^{s-1}}\\ &+\left(\mu_n|D(v_n)|^2,\partial_t b_n\right)_{H^{s-1}}.\end{aligned}$$

Using the Hölder and Young inequalities we get

$$\|\partial_t b_n\|_{H^{s-1}}^2 \leqslant C\bigg(\|v_n\nabla b_n\|_{H^{s-1}}^2+\|\nabla\cdot(\mu_n\nabla b_n)\|_{H^{s-1}}^2+\|b_n\omega_n\|_{H^{s-1}}^2+\left\|\mu_n|D(v_n)|^2\right\|_{H^{s-1}}^2\bigg).$$

By Lemmas 1.3.7 and 1.3.8 it easily follows

$$\|\partial_t b_n\|_{H^{s-1}}^2 \leqslant C\bigg(\|v_n\nabla b_n\|_{H^s}^2+\|\mu_n\nabla b_n\|_{H^s}^2+\|b_n\omega_n\|_{H^s}^2+\left\|\mu_n|D(v_n)|^2\right\|_{H^{s-1}}^2\bigg). \tag{4.80}$$

The most troublesome term to estimate is the last one on the right-hand side, so first let us concentrate on it. Let us choose $\varepsilon \geqslant 0$ in the following way

$$\begin{cases} \varepsilon \in (0, \min\{s - \frac{d}{2}, 1\}) & \text{for } d = 2 \\ \varepsilon = 0 & \text{for } d \geqslant 3 \end{cases}. \tag{4.81}$$

Based on Lemma 1.3.1 we have

$$\left\| \mu_n |D(v_n)|^2 \right\|_{H^{s-1}} \leqslant C \left(\left\| J^{s-1} \mu_n \right\|_{\frac{d}{\varepsilon + \frac{d}{2} - 1}} \left\| |Dv_n|^2 \right\|_{\frac{d}{1-\varepsilon}} + \|\mu_n\|_\infty \left\| J^{s-1}(|Dv_n|^2) \right\|_2 \right).$$

By applying Lemma 1.3.1 to the last term on the right-hand side we get

$$\left\| \mu_n |D(v_n)|^2 \right\|_{H^{s-1}} \leqslant C \left(\left\| J^{s-1} \mu_n \right\|_{\frac{d}{\varepsilon + \frac{d}{2} - 1}} \|Dv_n\|_{\frac{d}{1-\varepsilon}} + \|\mu_n\|_\infty \left\| J^{s-1} Dv_n \right\|_2 \right) \|Dv_n\|_\infty .$$

Let us observe that based on Lemma 1.3.9 we have

$$\left\| J^{s-1} \mu_n \right\|_{\frac{d}{\varepsilon + \frac{d}{2} - 1}} \leqslant C \left\| J^{s-\varepsilon} \mu_n \right\|_2 , \tag{4.82}$$

$$\|Dv_n\|_{\frac{d}{1-\varepsilon}} \leqslant C \left\| J^{\frac{d}{2}+\varepsilon} v_n \right\|_2 . \tag{4.83}$$

Using the above estimates and Lemmas 1.3.4, 1.3.7 yields

$$\left\| \mu_n |D(v_n)|^2 \right\|_{H^{s-1}} \leqslant C \left(\left\| J^{s-\varepsilon} \mu_n \right\|_2 \left\| J^{\frac{d}{2}+\varepsilon} v_n \right\|_2 + \|\mu_n\|_{H^s} \|v_n\|_{H^s} \right) \|v_n\|_{H^{s+1}} .$$

In view of Lemma 1.3.8 we get

$$\left\| \mu_n |D(v_n)|^2 \right\|_{H^{s-1}} \leqslant C \|\mu_n\|_{H^s} \|v_n\|_{H^s} \|v_n\|_{H^{s+1}} .$$

Using the above result in (4.80) and Lemmas 1.3.2, 1.3.7 we obtain

$$\begin{aligned} \|\partial_t b_n\|_{H^{s-1}}^2 \leqslant C \Big(& \|v_n\|_{H^s}^2 \|b_n\|_{H^{s+1}}^2 + \|\mu_n\|_{H^s}^2 \|b_n\|_{H^{s+1}}^2 + \|b_n\|_{H^s}^2 \|\omega_n\|_{H^s}^2 \\ & + \|\mu_n\|_{H^s}^2 \|v_n\|_{H^s}^2 \|v_n\|_{H^{s+1}}^2 \Big). \end{aligned}$$

The right-hand side of the above expression is in $L^1(0,T)$ (due to (4.78), (4.79) and (4.72)). Thus the estimate was proven. To conclude the time derivative estimates are as follows

$$\|\partial_t v_n, \partial_t \omega_n, \partial_t b_n\|_{L^2(0,T,H^{s-1}(\mathbb{T}^d))} \leqslant C\left(b_{\min}, \omega_{\min}, \omega_{\max}, \|v_0, \omega_0, b_0\|_{H^s}, s, T\right) < \infty. \tag{4.84}$$

4.2.4. Passage to the limit in approximate system, regularity of solution

By estimates (4.78), (4.79) and (4.84) we can conclude existence of sub-sequence $\{n_k\}$ (relabelled as n) such that:

$$v_n \to v \qquad \text{weakly in } L^2(0,T;H^{s+1}_{\text{div}}(\mathbb{T}^d)), \tag{4.85}$$

$$\omega, b_n \to \omega, b \qquad \text{weakly in } L^2(0,T;H^{s+1}(\mathbb{T}^d)), \tag{4.86}$$

$$\partial_t v_n, \partial_t \omega, \partial_t b_n \to \partial_t v, \partial_t \omega, \partial_t b \qquad \text{weakly in } L^2(0,T;H^{s-1}(\mathbb{T}^d)), \tag{4.87}$$

$$v_n, \omega, b_n \to v, \omega, b \qquad \text{weakly} * \text{ in } L^\infty(0,T;H^{s}(\mathbb{T}^d)). \tag{4.88}$$

Additionally, from the Aubin-Lions lemma, it follows that

$$v_n, \omega, b_n \to v, \omega, b \qquad \text{strongly in } L^2(0,T;H^{s'+1}(\mathbb{T}^d)), \tag{4.89}$$

$$v_n, \omega, b_n \to v, \omega, b \qquad \text{strongly in } C([0,T];H^{s'}(\mathbb{T}^d)). \tag{4.90}$$

for all $s' < s$. It is easy to see that

$$\mu_n \to \mu = \frac{\Psi_t(b)}{\Phi_t(\omega)} \qquad \text{strongly in } C([0,T];H^{s'}(\mathbb{T}^d)) \tag{4.91}$$

holds for all $\frac{d}{2} < s' < s$. Indeed, we see that with the help of the triangle inequality and Lemma 1.3.2 we get

$$\begin{aligned}\|\mu_n - \mu\|_{H^{s'}} &= \left\| \frac{\Psi_t(b_n) - \Psi_t(b)}{\Phi_t(\omega_n)} - \Psi_t(b)\frac{\Phi_t(\omega_n) - \Phi_t(\omega)}{\Phi_t(\omega)\Phi_t(\omega_n)} \right\|_{H^{s'}} \\ &\leqslant C\Bigg(\left\| \frac{1}{\Phi_t(\omega_n)} \right\|_{H^{s'}} \|\Psi_t(b_n) - \Psi_t(b)\|_{H^{s'}} \\ &\quad + \|\Psi_t(b)\|_{H^{s'}} \left\| \frac{1}{\Phi_t(\omega_n)} \right\|_{H^{s'}} \left\| \frac{1}{\Phi_t(\omega)} \right\|_{H^{s'}} \|\Phi_t(\omega_n) - \Phi_t(\omega)\|_{H^{s'}} \Bigg).\end{aligned}$$

From (4.71) we see that $\sup_{n\in\mathbb{N}} \operatorname{ess\,sup}_{t\in[0,T]} \left\| \frac{1}{\Phi_t(\omega_n(t))} \right\|_{H^{s'}}$ is finite. Furthermore, using the same reasoning as presented in (4.70), (4.71) we can conclude that $\operatorname{ess\,sup}_{t\in[0,T]} \left\| \frac{1}{\Phi_t(\omega(t))} \right\|_{H^{s'}}$

and ess $\sup_{t\in[0,T]} \|\Psi_t(b(t))\|_{H^{s'}}$ are finite. Thus, to prove (4.91) it is sufficient to show that

$$\Phi_t(\omega_n), \Psi_t(b_n) \to \Phi_t(\omega), \Psi_t(b) \text{ strongly in } C(0,T,H^{s'}(\mathbb{T}^d)).$$

This, however, holds based on (4.90), (4.9), (4.11) and Lemma 1.3.6.
Having convergence results, we may pass to the limit in (4.38) - (4.40). It is easy to see that v, ω, b satisfy:

$$(\partial_t v, w) + (v\cdot\nabla v, w) + (\mu Dv, Dw) = 0 \qquad \forall w \in H^1_{\text{div}}(\mathbb{T}^d), \quad (4.92)$$

$$(\partial_t \omega, z) + (v\cdot\nabla \omega, z) + (\mu \nabla\omega, \nabla z) = -\kappa_2\left(\omega^2, z\right) \qquad \forall z \in H^1(\mathbb{T}^d), \quad (4.93)$$

$$(\partial_t b, q) + (v\cdot\nabla b, q) + (\mu \nabla b, \nabla q) = -\left(b\omega, q\right) + \left(\mu|Dv|^2, q\right) \qquad \forall q \in H^1(\mathbb{T}^d) \quad (4.94)$$

for a.a. $t\in(0,T)$. We will provide more details for the most troublesome term. First, we wish to establish the convergence

$$\int_0^T \left(\mu_n|Dv_n|^2, \psi\right) dt \overset{n\to\infty}{\longrightarrow} \int_0^T \left(\mu|Dv|^2, \psi\right) dt, \qquad (4.95)$$

where $\psi \in L^2(0,T,H^1(\mathbb{T}^d))$. We see that

$$\left|\int_0^T \left(\mu_n|Dv_n|^2, \psi\right) dt - \int_0^T \left(\mu|Dv|^2, \psi\right) dt\right| \leqslant \left|\int_0^T \left(\mu_n\left(Dv_n - Dv\right)\left(Dv_n + Dv\right), \psi\right) dt\right| + \left|\int_0^T \left(\left(\mu_n - \mu\right)|Dv|^2, \psi\right) dt\right|.$$

Let us first focus on the first term of the right-hand side

$$\left|\int_0^T \left(\mu_n D(v_n - v) D(v_n + v), \psi\right) dt\right| \leqslant \int_0^T \|\mu_n\|_\infty \|D(v_n - v)\|_\infty \|D(v_n+v)\|_2 \|\psi\|_2\, dt \leqslant C\int_0^T \|\mu_n\|_{H^s} \|v_n - v\|_{H^{s'+1}} \|v_n+v\|_{H^{s'}} \|\psi\|_2\, dt,$$

where $s' \in \left(\frac{d}{2}, s\right)$. Using the Hölder inequality and (4.89), (4.90), (4.91) we get

$$\left|\int_0^T \left(\mu_n D(v_n - v) D(v_n + v), \psi\right) dt\right| \leqslant C\left(\int_0^T \|\mu_n\|^2_{H^{s'}} \|v_n+v\|^2_{H^{s'}} \|\psi\|_2^2\, dt\right)^{\frac{1}{2}} \|v_n - v\|_{L^2(0,T,H^{s'+1})} \overset{n\to\infty}{\longrightarrow} 0.$$

By using Lemma 1.3.4 we have

$$\left|\int_0^T \left((\mu_n - \mu)\,|Dv|^2, \psi\right) dt\right| \leqslant \int_0^T \|\mu_n - \mu\|_\infty \|Dv\|_\infty \|Dv\|_2 \|\psi\|_2 \, dt$$
$$\leqslant C \|\mu_n - \mu\|_{C(0,T,H^{s'})} \int_0^T \|v\|_{H^{s+1}} \|v\|_{H^s} \|\psi\|_2 \, dt.$$

Thus from (4.91), (4.88), (4.85) it follows

$$\left|\int_0^T \left((\mu_n - \mu)\,|Dv|^2, \psi\right) dt\right| \overset{n\to\infty}{\longrightarrow} 0.$$

Therefore (4.95) is proven.

Now we will show that

$$\omega^t_{\min} \leqslant \omega(t,x) \leqslant \omega^t_{\max} \quad \text{for a.e. } (x,t) \in \mathbb{T}^d \times [0,T] \tag{4.96}$$

and

$$b^t_{\min} \leqslant b \quad \text{for a.e. } (x,t) \in \mathbb{T}^d \times [0,T]. \tag{4.97}$$

The argument is similar to the ones presented in Chapter 2 or [8]. The main difference lies in the form of an approximate system i.e. the lack of certain cut-off functions in the current formulation. Thus, we present below the adapted reasoning. We denote by u_- (u_+) the positive (negative resp.) part of a function u. Then $u = u_+ + u_-$. We test the equation (4.93) by $(\omega - \omega^t_{\min})_-$ and obtain

$$\begin{aligned}(\omega_{,t}, (\omega - \omega^t_{\min})_-) + (v\nabla\omega, (\omega - \omega^t_{\min})_-) + \left(\mu\nabla\omega, \nabla\left(\omega - \omega^t_{\min}\right)_-\right)& \\ = -\kappa_2(\omega^2, (\omega - \omega^t_{\min})_-).&\end{aligned} \tag{4.98}$$

Using (4.3) we get

$$(\omega_{,t}, (\omega - \omega^t_{\min})_-) = \frac{1}{2}\frac{d}{dt}\|(\omega - \omega^t_{\min})_-\|_2^2 - \kappa_2\left((\omega^t_{\min})^2, (\omega - \omega^t_{\min})_-\right).$$

Hence, using that $0 < \mu$ and $\operatorname{div} v = 0$ in (4.98) we obtain

$$\frac{1}{2}\frac{d}{dt}\|(\omega - \omega^t_{\min})_-\|_2^2 - \kappa_2\left((\omega^t_{\min})^2, (\omega - \omega^t_{\min})_-\right) \leqslant -\kappa_2(\omega^2, (\omega - \omega^t_{\min})_-).$$

We write the above inequality in the form

$$\frac{1}{2}\frac{d}{dt}\|(\omega-\omega^t_{\min})_-\|_2^2 \leqslant -\kappa_2((\omega-\omega^t_{\min})(\omega+\omega^t_{\min}),(\omega-\omega^t_{\min})_-).$$

Based on (4.90) and Lemma 1.3.4 we have that $\omega \in L^\infty(0,T,L^\infty(\mathbb{T}^d))$. Thus from the above, we may derive the following estimate:

$$\frac{1}{2}\frac{d}{dt}\|(\omega-\omega^t_{\min})_-\|_2^2 \leqslant \kappa_2\left\|\omega+\omega^t_{\min}\right\|_{L^\infty(0,T,L^\infty(\mathbb{T}^d))}\|(\omega-\omega^t_{\min})_-\|_2^2.$$

Therefore, as $\|(\omega(0)-\omega^0_{\min})_-\|_2^2 = 0$ we obtain from Grönwall lemma that $\|(\omega(t)-\omega^t_{\min})_-\|_2^2 = 0\ \forall t \in [0,T]$. This implies the first inequality in (4.96).
If we test the equation (4.93) by $(\omega-\omega^t_{\max})_+$ then we obtain

$$\begin{aligned}(\omega_{,t},(\omega-\omega^t_{\max})_+) + (v\nabla\omega,(\omega-\omega^t_{\max})_+) + \left(\mu\nabla\omega,\nabla\left(\omega-\omega^t_{\max}\right)_+\right)\\ = -\kappa_2(\omega^2,(\omega-\omega^t_{\max})_+).\end{aligned} \tag{4.99}$$

Proceeding as before, we get

$$\frac{1}{2}\frac{d}{dt}\|(\omega-\omega^t_{\max})_+\|_2^2 - \kappa_2\left((\omega^t_{\max})^2,(\omega-\omega^t_{\max})_+\right) \leqslant -\kappa_2(\omega^2,(\omega-\omega^t_{\max})_+).$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(\omega-\omega^t_{\max})_+\|_2^2 &\leqslant -\kappa_2((\omega-\omega^t_{\max})(\omega+\omega^t_{\max}),(\omega-\omega^t_{\max})_+)\\ &= -\kappa_2((\omega+\omega^t_{\max}),|(\omega-\omega^t_{\max})_+|^2).\end{aligned}$$

Hence

$$\frac{1}{2}\frac{d}{dt}\|(\omega-\omega^t_{\max})_+\|_2^2 \leqslant 0. \tag{4.100}$$

Since $\|(\omega(0)-\omega^0_{\max})_+\|_2^2 = 0$ we obtain that $\|(\omega(t)-\omega^t_{\max})_+\|_2^2 = 0\ \forall t \in [0,T]$. This implies the second inequality in (4.96). Now we will show the non-negativity of b. We test the equation by (4.94) by b_- and we obtain

$$(b_{,t},b_-) + (v\nabla b,b_-) + (\mu\nabla b,\nabla b_-) = -(b\omega,b_-) + (\mu\left|D(v)\right|^2,b_-).$$

Using that $0 < \mu$, $\operatorname{div} v = 0$, we get

$$(\partial_t b_-, b_-) \leqslant -(b_- \omega, b_-).$$

Finally, using the lower bound from (4.96), we obtain

$$\frac{d}{dt}\|b_-\|_2^2 \leqslant 0.$$

Since $\|(b(0))_-\|_2^2 = 0$ we have that $\|(b(t))_-\|_2^2 = 0$ $\forall t \in [0,T]$. This implies the non-negativity of b.

Now we will show (4.97). For this purpose we test the equation (4.94) by $(b - b^t_{\min})_-$. Then we get

$$\begin{aligned}(b_{,t}, (b - b^t_{\min})_-) + (v\nabla b, (b - b^t_{\min})_-) + \big(\mu \nabla b, \nabla((b - b^t_{\min})_-)\big) &= -(b\omega, (b - b^t_{\min})_-) \\ &+ (\mu\, |D(v)|^2, (b - b^t_{\min})_-).\end{aligned} \tag{4.101}$$

The first term on the left-hand side is equal to

$$\frac{1}{2}\frac{d}{dt}\|(b - b^t_{\min})_-\|_2^2 - \left(\frac{\omega_{\max} b_{\min}}{(1 + \omega_{\max}\kappa_2 t)^{\frac{1}{\kappa_2}+1}}, (b - b^t_{\min})_- \right).$$

The second term of the left-hand side vanishes and the third one is non-negative. Thus

$$\frac{1}{2}\frac{d}{dt}\|(b - b^t_{\min})_-\|_2^2 - \left(\frac{\omega_{\max} b_{\min}}{(1 + \omega_{\max}\kappa_2 t)^{\frac{1}{\kappa_2}+1}}, (b - b^t_{\min})_- \right) \leqslant -(b\omega, (b - b^t_{\min})_-).$$

Using the upper bound from (4.96) and non-negativity of b, we obtain

$$\frac{1}{2}\frac{d}{dt}\|(b - b^t_{\min})_-\|_2^2 - \left(\frac{\omega_{\max} b_{\min}}{(1 + \omega_{\max}\kappa_2 t)^{\frac{1}{\kappa_2}+1}}, (b - b^t_{\min})_- \right) \leqslant -\frac{\omega_{\max}}{1 + \omega_{\max}\kappa_2 t}(b, (b - b^t_{\min})_-)$$

and by definition (4.3) we have

$$\frac{1}{2}\frac{d}{dt}\|(b - b^t_{\min})_-\|_2^2 \leqslant -\frac{\omega_{\max}}{1 + \omega_{\max}\kappa_2 t}\|(b - b^t_{\min})_-\|_2^2.$$

and then $\frac{d}{dt}\|(b - b^t_{\min})_-\|_2^2 \leqslant 0$. Since $\|(b(0) - b^0_{\min})_-\|_2^2 = 0$ we obtain that $\|(b(t) - b^t_{\min})_-\|_2^2 = 0 \ \forall t \in [0,T]$. This implies inequality (4.97).

Thus due to Definitions (4.11) and (4.9) and bounds (4.97), (4.96) we have

$$\mu = \frac{b}{\omega}$$

and thus (v,ω,b) solve system (1)-(4). Now we will show the continuity of the solution in $H^s\left(\mathbb{T}^d\right)$ norm. It is clear that $\left[H^{s-1}(\mathbb{T}^d), H^{s+1}(\mathbb{T}^d)\right]_{\frac{1}{2}} = H^s\left(\mathbb{T}^d\right)$. Thus from (4.86), (4.87) and the Lions-Magenes Lemma (see Theorem II.5.14 from [5]) we can conclude that

$$(v,\omega,b) \in C\left([0,T]; H^s\left(\mathbb{T}^d\right)\right).$$

The uniqueness of the obtained solution will easily follow from Theorem 4.1.2.

4.3. Proof of Theorem 4.1.2

First, in order to have proper bounds on viscosity term $\mu_i = \frac{b_i}{\omega_i}$, we would like to conclude that

$$\omega^t_{\min} \leqslant \omega_i(t,x) \leqslant \omega^t_{\max}, \quad b^t_{\min} \leqslant b_i(t,x) \quad \text{for a.e.} \quad (x,t) \in \mathbb{T}^d \times [0,T], \quad j = 1,2. \tag{4.102}$$

In view of the Lemma 1.3.4 we see that b_i and ω_i are continuous functions. Suppose that $\omega^t_{\min} > \omega_i(t^*,x^*)$ or $b^t_{\min} > b_i(t^*,x^*)$ for some $(t^*,x^*) \in [0,T] \times \mathbb{T}^d$ such that $\omega_i(t,x) > C^*_\omega > 0$, $b_i(t,x) > C^*_b > 0$ for all $(t,x) \in [0,t^*] \times \mathbb{T}^d$. Then, by following the procedures starting from (4.98) and (4.101) we obtain the contradiction. To find the upper bound on ω_i we proceed as in (4.99)-(4.100) (see also Proposition 3.3.1 in Chapter 3).

Let us denote $\delta_v = v_2 - v_1$, $\delta_\omega = \omega_2 - \omega_1$, $\delta_b = b_2 - b_1$. Then differences $(\delta_v,\delta_\omega,\delta_b)$ satisfy the following system of equations

$$\begin{aligned}(\partial_t \delta_v, w) + \left(\frac{b_1}{\omega_2} D\delta_v, Dw\right) &= -(v_2 \nabla \delta_v, w) - (\delta_v \nabla v_1, w) - \left(\frac{\delta_b}{\omega_2} Dv_2, Dw\right) \\ &\quad + \left(\frac{b_1 \delta_\omega}{\omega_1 \omega_2} Dv_1, Dw\right),\end{aligned} \tag{4.103}$$

$$
\begin{aligned}
(\partial_t \delta_\omega, z) + \left(\frac{b_1}{\omega_2}\nabla\delta_\omega, \nabla z\right) &= -(v_2\nabla\delta_\omega, z) - (\delta_v\nabla\omega_1, z) - \left(\frac{\delta_b}{\omega_2}\nabla\omega_2, \nabla z\right) \\
&\quad + \left(\frac{b_1\delta_\omega}{\omega_1\omega_2}\nabla\omega_1, \nabla z\right) - \kappa_2(\delta_\omega(\omega_2+\omega_1), z),
\end{aligned} \tag{4.104}
$$

$$
\begin{aligned}
(\partial_t \delta_b, z) + \left(\frac{b_1}{\omega_2}\nabla\delta_b, \nabla z\right) &= -(v_2\nabla\delta_b, z) - (\delta_v\nabla b_1, z) - \left(\frac{\delta_b}{\omega_2}\nabla b_2, \nabla z\right) \\
&\quad + \left(\frac{b_1\delta_\omega}{\omega_1\omega_2}\nabla b_1, \nabla z\right) - (\delta_b\omega_2, z) - (b_1\delta_\omega, z) \\
&\quad + \left(\frac{\delta_b}{\omega_2}|Dv_2|^2, z\right) + \left(\frac{b_1}{\omega_2}D\delta_v(Dv_2+Dv_1), z\right) \\
&\quad - \left(\frac{b_1\delta_\omega}{\omega_2\omega_1}|Dv_1|^2, z\right).
\end{aligned} \tag{4.105}
$$

Now, we test equations (4.103)-(4.105) with δ_v, δ_ω, δ_b to obtain an estimate of differences in the L^2 norm. We will show the procedure for δ_b, since the rest are analogous

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\delta_b\|_2^2 + \left(\frac{b_1}{\omega_2}\nabla\delta_b, \nabla\delta_b\right) &= -(v_2\nabla\delta_b, \delta_b) - (\delta_v\nabla b_1, \delta_b) - \left(\frac{\delta_b}{\omega_2}\nabla b_2, \nabla\delta_b\right) \\
&+ \left(\frac{b_1\delta_\omega}{\omega_1\omega_2}\nabla b_1, \nabla\delta_b\right) - (\delta_b\omega_2, \delta_b) - (b_1\delta_\omega, \delta_b) + \left(\frac{\delta_b}{\omega_2}|Dv_2|^2, \delta_b\right) \\
&+ \left(\frac{b_1}{\omega_2}D\delta_v(Dv_2+Dv_1), \delta_b\right) - \left(\frac{b_1\delta_\omega}{\omega_2\omega_1}|Dv_1|^2, \delta_b\right).
\end{aligned}
$$

Using (4.102) and the Hölder inequality we get

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\delta_b\|_2^2 + \mu^t_{\min}\|\nabla\delta_b\|_2^2 &\leqslant \|\delta_v\|_2\|\nabla b_1\|_\infty\|\delta_b\|_2 + \left\|\frac{1}{\omega_2}\right\|_\infty\|\delta_b\|_2\|\nabla b_2\|_\infty\|\nabla\delta_b\|_2 \\
&+ \|b_1\|_\infty\|\delta_\omega\|_2\left\|\frac{1}{\omega_1\omega_2}\right\|_\infty\|\nabla b_1\|_\infty\|\nabla\delta_b\|_2 + \|b_1\|_\infty\|\delta_\omega\|_2\|\delta_b\|_2 \\
&+ \left\|\frac{1}{\omega_2}\right\|_\infty\|Dv_2\|_\infty^2\|\delta_b\|_2^2 + \left\|\frac{b_1}{\omega_2}\right\|_\infty\|D\delta_v\|_2\|Dv_2+Dv_1\|_\infty\|\delta_b\|_2 \\
&+ \left\|\frac{1}{\omega_2\omega_1}\right\|_\infty\|b_1\|_\infty\|Dv_1\|_\infty^2\|\delta_\omega\|_2\|\delta_b\|_2.
\end{aligned}
$$

By (4.102) and Lemma 1.3.4 we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\delta_b\|_2^2 + \mu^t_{\min}\|\nabla\delta_b\|_2^2 &\leqslant \|b_1\|_{H^{s+1}}\left(\|\delta_v\|_2^2 + \|\delta_b\|_2^2\right) + C(\omega^t_{\min})\|b_2\|_{H^{s+1}}^2\|\delta_b\|_2^2 \\
&+ \frac{\mu^t_{\min}}{6}\|\nabla\delta_b\|_2^2 + C(\omega^t_{\min})\|b_1\|_{H^s}^2\|b_1\|_{H^{s+1}}^2\|\delta_\omega\|_2^2 + \frac{\mu^t_{\min}}{6}\|\nabla\delta_b\|_2^2 \\
&+ \|b_1\|_{H^s}\left(\|\delta_\omega\|_2^2 + \|\delta_b\|_2^2\right) + C(\omega^t_{\min})\|v_2\|_{H^{s+1}}^2\|\delta_b\|_2^2 \\
&+ C(\omega^t_{\min})\|b_1\|_{H^s}^2\left(\|v_2\|_{H^{s+1}}^2 + \|v_1\|_{H^{s+1}}^2\right)\|\delta_b\|_2^2 + \frac{\mu^t_{\min}}{3}\|D\delta_v\|_2^2
\end{aligned}
$$

$$+ C(\omega^t_{\min}) \|b_1\|_{H^s} \|v_1\|^2_{H^{s+1}} \left(\|\delta_\omega\|_2^2 + \|\delta_b\|_2^2\right) .$$

Thus the information about the regularity of the solutions yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\delta_b\|_2^2 + \mu^t_{\min}\|\nabla\delta_b\|_2^2 \leqslant G_b(t) \left(\|\delta_v\|_2^2 + \|\delta_\omega\|_2^2 + \|\delta_b\|_2^2\right) \\ + \frac{\mu^t_{\min}}{3}\left(\|\nabla\delta_v\|_2^2 + \|\nabla\delta_\omega\|_2^2 + \|\nabla\delta_b\|_2^2\right) ,\end{aligned} \tag{4.106}$$

where $G_b \in L^1(0,T)$. Analogously, there exist functions $G_\omega, G_v \in L^1(0,T)$ such that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\delta_\omega\|_2^2 + \mu^t_{\min}\|\nabla\delta_\omega\|_2^2 \leqslant G_\omega(t) \left(\|\delta_v\|_2^2 + \|\delta_\omega\|_2^2 + \|\delta_b\|_2^2\right) \\ + \frac{\mu^t_{\min}}{3}\left(\|\nabla\delta_v\|_2^2 + \|\nabla\delta_\omega\|_2^2 + \|\nabla\delta_b\|_2^2\right) ,\end{aligned} \tag{4.107}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\delta_v\|_2^2 + \mu^t_{\min}\|\nabla\delta_v\|_2^2 \leqslant G_v(t) \left(\|\delta_v\|_2^2 + \|\delta_\omega\|_2^2 + \|\delta_b\|_2^2\right) \\ + \frac{\mu^t_{\min}}{3}\left(\|\nabla\delta_v\|_2^2 + \|\nabla\delta_\omega\|_2^2 + \|\nabla\delta_b\|_2^2\right) .\end{aligned} \tag{4.108}$$

Summing (4.108), (4.107) and (4.106) we get

$$\frac{1}{2}\frac{d}{dt}\left(\|\delta_v\|_2^2 + \|\delta_\omega\|_2^2 + \|\delta_b\|_2^2\right) \leqslant G(t)\left(\|\delta_v\|_2^2 + \|\delta_\omega\|_2^2 + \|\delta_b\|_2^2\right) ,$$

where $G = G_v + G_\omega + G_b \in L^1(0,T)$. From the Grönwall inequality it follows that $\delta_v(t,x) = 0$, $\delta_\omega(t,x) = 0$, $\delta_b(t,x) = 0$ for a.a. $(x,t) \in \mathbb{T}^d \times [0,T]$. This concludes the proof of the uniqueness.

Chapter 5

Existence of a weak solution

In this chapter, the existence of a global weak solution to Kolmogorov's model of turbulence on torus will be shown. Recently in [8] the analogous result was shown in the case of bounded domains. The used methodology exhibits an additional layer of complication due to the imposed boundary conditions. To better understand the important steps in the developed approach, the chosen domain is the torus. Chapter's contents follow the considerations of [8], providing additional justifications when needed. The chapter is based on [28].

5.1. Formulation of the theorem

Assume that $\Omega = \prod_{i=1}^{3}(0, 2\pi)$, $T > 0$ and $\Omega^T = \Omega \times (0, T)$. Here, $\nu_0, \kappa_1, \ldots, \kappa_4$ are positive constants. For simplicity, we assume that all constants except of κ_2 are equal to one. The reason is that κ_2 plays an important role in a priori estimates.
Now, we specify the initial data. Let us assume in a standard way, that the initial condition for the velocity field fulfils

$$v_0 \in L^2_{\mathrm{div}}(\Omega). \tag{5.1}$$

For the turbulent kinetic energy, we assume that initially it is as follows:

$$b_0 \in L^1(\Omega), \qquad \ln b_0 \in L^1(\Omega), \qquad b_0 > 0. \tag{5.2}$$

Finally, the initial values of the dissipation ω are as follows:

$$\omega_0 \in L^\infty(\Omega), \qquad 0 < \omega_{\min} \leqslant \omega_0 \leqslant \omega_{\max} < \infty. \tag{5.3}$$

Now, we are ready to present the main theorem, which states the existence result to the system (1)-(5).

Theorem 5.1.1. *Let us assume that the initial data satisfy (5.1)-(5.3). Then, there exists a quadruple* (v, b, ω, p) *such that*

$$v \in L^2(0,T,W^{1,2}_{\mathrm{div}}(\Omega)) \cap W^{1,q}(0,T,W^{-1,q}(\Omega)) \qquad \text{for all } q \in \left[1, \frac{16}{11}\right), \tag{5.4}$$

$$b \in \varepsilon, \tag{5.5}$$

$$\partial_t b \in \mathcal{M}\left(0,T,W^{-1,q}(\Omega)\right) \qquad \text{for all } q \in \left[1, \frac{8}{7}\right), \tag{5.6}$$

$$p \in L^q\left(0,T,L^q_0(\Omega)\right) \qquad \text{for all } q \in \left[1, \frac{16}{11}\right), \tag{5.7}$$

$$E \in W^{1,q}\left(0,T,W^{-1,q}(\Omega)\right) \qquad \text{for all } q \in \left[1, \frac{80}{79}\right), \tag{5.8}$$

$$b\omega \in L^q(0,T,W^{1,q}(\Omega)) \qquad \text{for all } q \in \left[1, \frac{16}{11}\right), \tag{5.9}$$

$$\frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t} \leqslant \omega \leqslant \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t} \qquad \text{almost everywhere in } \Omega^T, \tag{5.10}$$

where, ε *was defined in (1.2) and*

$$E := \frac{|v|^2}{2} + b. \tag{5.11}$$

In addition, the pressure p *can be decomposed as* $p = p_1 + p_2$*, where*

$$p_1 \in L^q\left(0,T,L^q_0(\Omega)\right) \text{ for all } q \in \left[1, \frac{16}{11}\right), \tag{5.12}$$

$$p_2 \in L^{5/3}\left(0,T,L^{5/3}_0(\Omega)\right). \tag{5.13}$$

After denoting

$$\mu := \frac{b}{\omega}, \tag{5.14}$$

the quadruple (v, b, ω, p) satisfies the following identities:

$$\int_0^T \langle v_{,t}, w\rangle - (v \otimes v, \nabla w) + (\mu D(v), D(w))\, dt = \int_0^T (p, \operatorname{div} w)\, dt \qquad \forall w \in L^\infty\left(0, T, W^{1,\infty}(\Omega)\right), \tag{5.15}$$

$$\int_0^T \langle \partial_t E, z\rangle - (v\,(E + p)\,, \nabla z) + (\mu \nabla b, \nabla z) + (\mu D(v) v, \nabla z)\, dt = -\int_0^T (b\omega, z)\, dt \qquad \forall z \in L^\infty\left(0, T, W^{1,\infty}(\Omega)\right), \tag{5.16}$$

$$\int_0^T \langle \partial_t \omega, z\rangle - (v\omega, \nabla z) + \left(\frac{\nabla\,(b\omega)}{\omega} - \nabla b, \nabla z\right) dt = -\kappa_2 \int_0^T \left(\omega^2, z\right) dt \qquad \forall z \in L^\infty\left(0, T, W^{1,\infty}(\Omega)\right), \tag{5.17}$$

with the initial data fulfilling

$$\lim_{t \to 0^+} \|v(t) - v_0\|_2 + \|\omega(t) - \omega_0\|_2 + \|b(t) - b_0\|_1 = 0. \tag{5.18}$$

Moreover, the following inequality holds:

$$\int_0^T \langle b_{,t}, z\rangle + (\mu \nabla b, \nabla z) - (vb, \nabla z)\, z dt \geqslant \int_0^T \left(-b\omega + \mu |D(v)|^2, z\right) dt \qquad \forall z \in C\left(0, T, W^{1,\infty}(\Omega)\right) \quad \textit{such that} \quad z \geqslant 0 \quad \textit{almost everywhere in} \quad \Omega^T. \tag{5.19}$$

In order to prove the above result, we will establish several existence results to auxiliary problems, which approximate the problem (1)-(5). Using established estimates in those approximations, it will be plausible to obtain the existence result of the considered system. Now, we will focus on outlining auxiliary lemmas and notation.

5.2. Proof of Theorem 5.1.1 and auxiliary Theorems

The proof will be divided into subsections to provide a more transparent view of each step. Firstly, the additional notation and facts relevant to subsequent considerations will be provided. Next, the solutions to approximated systems will be obtained.

5.2.1. Auxiliary results and additional notation

To define approximate problems, we introduce the cut-off function

$$T_m(s) = \begin{cases} s & \text{if } |s| \leqslant m \\ m\,\mathrm{sgn}(s) & \text{if } |s| > m \end{cases}. \tag{5.20}$$

Now, we define the function Θ_m, which is the primitive function of T_m:

$$\Theta_m(s) := \int_0^s T_m(\tau)d\tau. \tag{5.21}$$

Next, we consider a smooth, non-increasing function G, such that $G(s) = 1$ when $s \in [0,1]$ and $G(s) = 0$ for $s \geqslant 2$. For $m \in \mathbb{R}_+$, we define

$$G_m(s) := G\left(\frac{s}{m}\right) \tag{5.22}$$

and we denote

$$\Gamma_m(s) := \int_0^s G_m(\tau)d\tau. \tag{5.23}$$

In order to avoid the confusion, we define $z_+ = \max\{z, 0\}$ and $z_- = \min\{z, 0\}$.

Additionally, by $\{w_i\}_{i=0}^{\infty}$ we denote an orthogonal basis of $W^{1,2}_{\mathrm{div}}(\Omega)$, which is also orthogonal in L^2_{div} (such a basis exists due to Lemma 5.2.5). By $\{z_i\}_{i=0}^{\infty}$, we denote an orthogonal basis of $W^{1,2}(\Omega)$, which is also an orthogonal in $L^2(\Omega)$.

In the proof of the main theorem we will need to reconstruct the pressure. The following lemma will enable us to do so:

Lemma 5.2.1 (see Lemma C.1 in [6]). *Let $q, q' \in (1,\infty)$, and such that $\frac{1}{q} + \frac{1}{q'} = 1$. Then, there exists a linear, bounded operator*

$$\mathcal{L} : L^q(\Omega)^{3\times 3} \to L^q(\Omega), \tag{5.24}$$

such that for all $\varphi \in W^{2,q'}(\Omega)$ and any fixed $B \in L^q(\Omega)^{3\times 3}$ the following relation holds:

$$(\mathcal{L}(B), \Delta\varphi) = \left(B, \nabla^2\varphi\right), \qquad \int_\Omega \mathcal{L}(B)dx = 0. \tag{5.25}$$

Proof. For $B \in \mathcal{D}(\Omega)^{3\times 3}$ we set the system

$$\Delta\mathcal{L}(B) = \operatorname{div}\operatorname{div} B \quad \text{in } \Omega, \tag{5.26}$$

$$\int_\Omega \mathcal{L}(B)dx = 0 \tag{5.27}$$

equipped with the periodic boundary conditions. From the classical theory of the Poisson equation, the solution to system (5.26)-(5.27) exists and is smooth. Thus, we can write $\mathcal{L}(B) := (\Delta)^{-1}\operatorname{div}\operatorname{div} B$. Operator $\mathcal{L}$ is linear and continuous as a mapping from $W^{1,q}(\Omega)^{3\times 3}$ to $W^{1,q}(\Omega)$ for all $q \in (1,\infty)$. We also see that multiplying the equation (5.26) by arbitrary $\varphi \in W^{2,q'}(\Omega)$, and integrating by parts four times, we get (5.25). Now, we focus on showing the boundedness of the operator $\mathcal{L} : L^q(\Omega)^{3\times 3} \cap \mathcal{D}(\Omega)^{3\times 3} \to L^q(\Omega)$. To do this, we need to find a space-periodic φ such that

$$\Delta\varphi = |\mathcal{L}(B)|^{q-2}\mathcal{L}(B) - \frac{1}{|\Omega|}\int_\Omega |\mathcal{L}(B)|^{q-2}\mathcal{L}(B)dx \quad \text{in } \Omega, \tag{5.28}$$

$$\int_\Omega \varphi dx = 0. \tag{5.29}$$

From the L^q theory for the Poisson equation, there exists a constant $C > 0$ depending only on Ω and q such that

$$\begin{aligned}\int_\Omega |\nabla^2\varphi|^{q'}dx &\leqslant C\int_\Omega \left||\mathcal{L}(B)|^{q-2}\mathcal{L}(B) - \frac{1}{|\Omega|}\int_\Omega |\mathcal{L}(B)|^{q-2}\mathcal{L}(B)dx\right|^{q'} dx \\ &\leqslant C\int_\Omega |\mathcal{L}(B)|^q dx,\end{aligned} \tag{5.30}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. Since B is smooth, the integral on the right-hand side is finite for any $q \in (1,\infty)$. Now, plugging (5.28) into (5.25) we get using of the fact that $\int_\Omega \mathcal{L}(B)dx = 0$ and (5.30), the following inequality:

$$\int_\Omega |\mathcal{L}(B)|^q dx = \left(B, \nabla^2\varphi\right) \leqslant \|B\|_q \left\|\nabla^2\varphi\right\|_{q'} \leqslant C\|B\|_q \|\mathcal{L}(B)\|_q^{q-1}.$$

We obtained $\|\mathcal{L}(B)\|_q \leqslant C\|B\|_q$ for $B \in \mathcal{D}(\Omega)^{3\times 3}$. Since $\mathcal{D}(\Omega)^{3\times 3}$ is a dense subset of $L^q(\Omega)^{3\times 3}$, the operator can be uniquely extended to $\mathcal{L} : L^q(\Omega)^{3\times 3} \to L^q(\Omega)$. Moreover, the system (5.25) can be established for $B \in L^q(\Omega)^{3\times 3}$ by considering a sequence of smooth

$\{B^n\}$ such that $B^n \to B$ in $L^q(\Omega)^{3\times 3}$ and using the weak convergence. This completes the proof. □

For the completeness of the presented arguments, we recall the Div-Curl lemma.

Lemma 5.2.2 (see:[44, 37]). *Let Ω be an open set of R^N, $N \geqslant 2$. Let w be a function such that $w : \mathbb{R}^N \to \mathbb{R}$. We denote*

$$\mathrm{div}(w) = \sum_{i=1}^{n} \frac{\partial w_i}{\partial x_i}, \qquad C_{ij}(w) = \frac{\partial w_i}{\partial x_j} - \frac{\partial w_j}{\partial x_i}.$$

Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. For any n, let $a^n \in [L^p(\Omega)]^N$, $b^n \in [L^q(\Omega)]^N$ with the properties

$$a^n \overset{n\to\infty}{\rightharpoonup} a \quad \textit{weakly in} \, [L^p(\Omega)]^N,$$

$$b^n \overset{n\to\infty}{\rightharpoonup} b \quad \textit{weakly in} \, [L^q(\Omega)]^N,$$

$$\{\mathrm{div}\,(a^n)\}_{n=0}^{\infty} \quad \textit{lies in a compact subset of } W^{-1,p}(\Omega), \tag{5.31}$$

$$\{C\,(b^n)\}_{n=0}^{\infty} \quad \textit{lies in a compact subset of } W^{-1,q}(\Omega)^{N\times N}. \tag{5.32}$$

Then,

$$a^n b^n \overset{n\to\infty}{\rightharpoonup} ab \quad \textit{in the sense of distributions.}$$

Now, let us formulate a simple corollary of the Vitali convergence lemma.

Lemma 5.2.3 (see Corollary 4.5.5 in [4]). *Let $\Omega \subset \mathbb{R}^N$ be bounded and $u_n : \Omega \to \mathbb{R}$ be a sequence in $L^p(\Omega)$ for some $p > 1$. Suppose that*

1. *$u_n \to u$ almost everywhere in Ω,*
2. *the sequence u_n is bounded in $L^p(\Omega)$.*

Then,

$$u_n \to u \qquad \textit{in } L^r(\Omega) \textit{ for all } 1 \leqslant r < p.$$

Lemma 5.2.4. *Let $p, q \in (1, \infty)$ such that $\frac{1}{q} + \frac{1}{p} < 1$. Moreover, we assume that*

$$u_n \rightharpoonup u \quad \text{weakly in } L^p(\Omega) \qquad \text{and} \qquad v_n \to v \quad \text{strongly in } L^q(\Omega).$$

Then,

$$u_n v_n \rightharpoonup uv \quad \text{weakly in } L^s(\Omega),$$

where $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$.

Proof. Let s' be such that $\frac{1}{s'} + \frac{1}{s} = 1$. We see that $\frac{1}{s'} + \frac{1}{p} + \frac{1}{q} = 1$. Additionally, let $\varphi \in L^{s'}(\Omega)$. Then we have

$$\int_\Omega u_n v_n \varphi dx = \int_\Omega u_n (v_n - v) \varphi dx + \int_\Omega u_n v \varphi dx.$$

The first integral's limit is zero due to the strong convergence of v_n, boundedness of u_n and the following inequality:

$$\left| \int_\Omega u_n (v_n - v) \varphi dx \right| \leqslant \|u_n\|_p \|v_n - v\|_q \|\varphi\|_{s'} \xrightarrow{n\to\infty} 0.$$

Due to the fact that $v\varphi \in L^{p'}(\Omega)$, where $\frac{1}{p'} = \frac{1}{s'} + \frac{1}{q}$, and the weak convergence of u_n, we have

$$\int_\Omega u_n v \varphi dx \to \int_\Omega uv\varphi dx.$$

This completes the proof of the lemma. □

Lemma 5.2.5 (see: Theorem 2.24 in [41])**.** *There exists a family of functions $\mathcal{N} = \{a_1, a_2, a_3, \dots\}$ such that*

— *$\mathcal{N}$ is an orthonormal basis in $L^2_{\text{div}}(\Omega)$,*

— *$a_j \in C^\infty(\overline{\Omega})$,*

— *$\mathcal{N}$ is an orthogonal basis in $W^{1,2}_{\text{div}}(\Omega)$.*

Lemma 5.2.6 (Aubin–Lions–Simon, see: Theorem II.5.16 in [5])**.** *Let $B_0 \subset B_1 \subset B_2$ be three Banach spaces. We assume that the embedding of B_1 in B_2 is continuous and that* *the embedding of B_0 in B_1 is compact. Let p, r be such that $1 \leqslant p, r \leqslant \infty$. For $T > 0$, we define*

$$E_{p,r} = \left\{ v \in L^p\left(0, T, B_0\right), \; \frac{dv}{dt} \in L^r\left(0, T, B_2\right) \right\}.$$

i) If $p < \infty$, the embedding of $E_{p,r}$ in $L^p\left(0, T, B_1\right)$ is compact.

ii) If $p = \infty$ an if $r > 1$, the embedding of $E_{p,r}$ in $C^0\left(0, T, B_1\right)$ is compact.

Lemma 5.2.7 (see: Chapter 1.2.b in [25])**.** *Let X be a Banach space, $T > 0$ and $1 \leqslant p \leqslant \infty$. Let $f_n \to f$ in $L^p(0, T, X)$. Then, there exists a subsequence f_{n_k} such that $f_{n_k} \to f$ in X almost everywhere.*

5.2.2. k-approximation

In order to prove Theorem 5.1.1, we will establish a series of existence results to approximate problems. We consider the following problem:

$$v_{,t} + \operatorname{div}(G_k\left(|v|^2\right) v \otimes v) - \operatorname{div}\left(T_k\left(\mu\right) D(v)\right) = -\nabla p, \tag{5.33}$$

$$\omega_{,t} + \operatorname{div}(\omega v) - \operatorname{div}\left(\frac{b}{\omega}\nabla\omega\right) = -\kappa_2\omega^2, \tag{5.34}$$

$$b_{,t} + \operatorname{div}(bv) - \operatorname{div}\left(\frac{b}{\omega}\nabla b\right) = -b\omega + T_k(\mu)|D(v)|^2, \tag{5.35}$$

$$\operatorname{div} v = 0, \tag{5.36}$$

in Ω^T, where $\mu = \frac{b}{\omega}$. The system is equipped with the periodic boundary conditions and the following initial condition:

$$v_{|t=0} = v_0, \quad \omega_{|t=0} = \omega_0, \quad b_{|t=0} = b_0^k(x) = b_0(x) + \frac{1}{k}. \tag{5.37}$$

The following theorem states the existence result for this system:

Theorem 5.2.8. *Let us fix $k \in \mathbb{N}_+$. Then, there exists a triple (v, b, ω) such that*

$$v \in L^2(0,T,W^{1,2}_{\mathrm{div}}(\Omega)) \cap W^{1,2}(0,T,W^{-1,2}_{\mathrm{div}}(\Omega)), \tag{5.38}$$

$$b \in L^q\left(0,T,W^{1,q}(\Omega)\right) \cap L^\infty\left(0,T,L^1(\Omega)\right) \qquad \textit{for all } q \in \left[1, \frac{5}{4}\right), \tag{5.39}$$

$$\partial_t b \in L^1\left(0,T,W^{-1,q}(\Omega)\right) \qquad \textit{for all } q \in \left[1, \frac{80}{79}\right), \tag{5.40}$$

$$\omega \in L^2\left(0,T,W^{1,2}(\Omega)\right) \cap L^\infty\left(0,T,L^\infty(\Omega)\right), \tag{5.41}$$

$$\partial_t \omega \in L^q\left(0,T,W^{-1,q}(\Omega)\right) \qquad \textit{for all } q \in \left[1, \frac{16}{11}\right), \tag{5.42}$$

$$\frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t} \leqslant \omega \leqslant \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t} \qquad \textit{almost everywhere in } \Omega^T, \tag{5.43}$$

$$\frac{1}{k\left(1+\kappa_2\omega_{\min}t\right)^{\frac{1}{\kappa_2}}} \leqslant b, \qquad \textit{almost everywhere in } \Omega^T, \tag{5.44}$$

which solves the problem (5.33)-(5.37) in the following sense:

$$\begin{aligned}\langle v_{,t}, w\rangle - \left(G_k\left(|v|^2\right) v\otimes v, \nabla w\right) + \left(T_k(\mu) D(v), D(w)\right) = 0 \\ \forall w \in W^{1,2}_{\mathrm{div}}(\Omega) \ \textit{a.a. } t \in (0,T),\end{aligned} \tag{5.45}$$

$$\begin{aligned}\langle b_{,t}, z\rangle - (bv, \nabla z) + (\mu\nabla b, \nabla z) = \left(-b\omega + T_k(\mu)|D(v)|^2, z\right) \\ \forall z \in W^{1,\infty}(\Omega) \ \textit{a.a. } t \in (0,T),\end{aligned} \tag{5.46}$$

$$\begin{aligned}\langle \omega_{,t}, z\rangle - (\omega v, \nabla z) + (\mu\nabla\omega, \nabla z) = -\kappa_2\left(\omega^2, z\right) \\ \forall z \in W^{1,\infty}(\Omega) \ \textit{a.a. } t \in (0,T),\end{aligned} \tag{5.47}$$

where $\mu = \frac{b}{\omega}$. The initial data are attained strongly in the following sense:

$$\lim_{t\to 0^+} \|v(t)-v_0\|_2 + \|\omega(t)-\omega_0\|_2 + \|b(t)-b_0^k\|_1 = 0. \tag{5.48}$$

Moreover, for all $\lambda \in (0,1]$, the following (k-independent) estimate holds:

$$\begin{aligned}&\sup_{t\in(0,T)}\left(\|b(t)\|_1 + \|\ln b(t)\|_1 + \|v(t)\|_2^2\right) + \int_{\Omega^T}\left(1+b^{-1}\right)T_k(\mu)|D(v)|^2 dxdt \\ &\quad + \int_{\Omega^T}\frac{\mu}{b^{1+\lambda}}|\nabla b|^2 + \mu|\nabla\omega|^2 + \mu^{\frac{8}{3}-\lambda}dxdt \\ &\leqslant C(\lambda^{-1}, v_0, b_0, \omega_0, \omega_{\min}, \omega_{\max}).\end{aligned} \tag{5.49}$$

Moreover, the following inequality holds for almost all times $t \in (0,T)$:

$$\begin{aligned} \left(\sqrt{b(t)}, \varphi\right) - \int_0^t \left(\sqrt{b}v, \nabla\varphi\right) d\tau + \int_0^t \left(\frac{\sqrt{b}}{2\omega}\nabla b, \nabla\varphi\right) d\tau \\ \geqslant \frac{1}{2}\int_0^t \left(\sqrt{b}\omega, \varphi\right) d\tau + \left(\sqrt{b_0^k}, \varphi\right) \qquad \forall \varphi \in D(\Omega), \varphi \geqslant 0. \end{aligned} \tag{5.50}$$

5.2.3. (n,k)-approximation

In order to prove Theorem 5.2.8, we introduce the next level of approximation. To define this approximate problem, we first smooth out the initial conditions for b and v. First, we find a sequence of smooth, non-negative functions b_0^n such that

$$b_0^n \to b_0 \text{ strongly in } L^1(\Omega) \tag{5.51}$$

and define

$$b_0^{n,k} := b_0^n + \frac{1}{k}. \tag{5.52}$$

Now, let $\{w_i\}_{i=0}^{\infty}$ be a smooth basis of $W^{1,2}_{\mathrm{div}}(\Omega)$ that is orthonormal in $L^2(\Omega)$. It exists due to Lemma 5.2.5. Using the chosen basis, we introduce the approximated initial condition for velocity v^n as

$$v_0^n = \sum_{i=1}^{n} (v_0, w_i) w_i.$$

Having approximated initial conditions, we consider the following problem:

$$v_{,t} + \mathrm{div}(G_k\left(|v|^2\right) v \otimes v) - \mathrm{div}\left(T_k\left(\mu^n\right) D(v)\right) = -\nabla p, \tag{5.53}$$

$$\omega_{,t} + \mathrm{div}(\omega v) - \mathrm{div}\left(T_n\left(\mu^n\right) \nabla\omega\right) = -\kappa_2\omega^2, \tag{5.54}$$

$$b_{,t} + \mathrm{div}(bv) - \mathrm{div}\left(T_n\left(\mu^n\right) \nabla b\right) = -b\omega + \frac{T_k(\mu^n)|D(v)|^2}{1 + n^{-1}|D(v)|^2}, \tag{5.55}$$

$$\mathrm{div}\, v = 0 \tag{5.56}$$

in Ω^T, where

$$\mu^n = \frac{b}{\omega} + \frac{1}{n}. \tag{5.57}$$

Additionally, the problem is equipped with the periodic boundary conditions and the following initial condition:

$$v_{|t=0} = v_0^n, \quad \omega_{|t=0} = \omega_0, \quad b_{|t=0} = b_0^{n,k}(x). \tag{5.58}$$

The following theorem states the existence result for this system:

Theorem 5.2.9. *Let us fix $k \in \mathbb{N}_+$, $n \in \mathbb{N}_+$. Then, there exists a triple (c, b, ω) such that*

$$c \in W^{1,\infty}(0,T)^n, \tag{5.59}$$

$$b \in L^2\left(0,T,W^{1,2}(\Omega)\right) \cap L^\infty\left(0,T,L^2(\Omega)\right), \tag{5.60}$$

$$\partial_t b \in L^2\left(0,T,W^{-1,2}(\Omega)\right), \tag{5.61}$$

$$\omega \in L^2\left(0,T,W^{1,2}(\Omega)\right) \cap L^\infty\left(0,T,L^\infty(\Omega)\right), \tag{5.62}$$

$$\partial_t \omega \in L^2\left(0,T,W^{-1,2}(\Omega)\right), \tag{5.63}$$

$$\frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t} \leqslant \omega \leqslant \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t} \qquad \textit{almost everywhere in } \Omega^T, \tag{5.64}$$

$$\frac{1}{k\left(1+\kappa_2\omega_{\min}t\right)^{\frac{1}{\kappa_2}}} \leqslant b \qquad \textit{almost everywhere in } \Omega^T, \tag{5.65}$$

which solves the problem (5.53)-(5.58) in the following sense:

$$(v_{,t}, w_i) - \left(G_k\left(|v|^2\right) v\otimes v, \nabla w_i\right) + \left(T_k\left(\mu^n\right) D(v), D(w_i)\right) = 0 \quad \textit{for all } i = 1,\dots,n, \tag{5.66}$$

$$\langle b_{,t}, z\rangle - (bv,\nabla z) + (T_n(\mu^n)\nabla b,\nabla z) + (b\omega, z) = \left(\frac{T_k\left(\mu^n\right)|D(v)|^2}{1+n^{-1}|D(v)|^2}, z\right) \quad \forall z \in W^{1,2}(\Omega)\, a.a.\ t\in(0,T), \tag{5.67}$$

$$\langle \omega_{,t}, z\rangle - (\omega v,\nabla z) + (T_n(\mu^n)\nabla\omega,\nabla z) = -\kappa_2\left(\omega^2, z\right) \quad \forall z \in W^{1,2}(\Omega)\, a.a.\ t\in(0,T), \tag{5.68}$$

where $\mu^n = \frac{b}{\omega} + \frac{1}{n}$ and

$$v(t,x) := \sum_{i=1}^{n} c_i(t) w_i(x). \tag{5.69}$$

The initial data are attained strongly in the following sense:

$$\lim_{t\to 0^+} \|v(t) - v_0^n\|_2 + \|\omega(t) - \omega_0\|_2 + \|b(t) - b_0^{n,k}\|_1 = 0. \tag{5.70}$$

Moreover, the following inequality holds for almost all times $t \in (0, T)$:

$$\begin{aligned} &\left(\sqrt{b(t)}, \varphi\right) - \int_0^t \left(\sqrt{b}v, \nabla\varphi\right) d\tau + \int_0^t \left(T_n(\mu^n)\nabla\sqrt{b}, \nabla\varphi\right) d\tau \\ &\qquad \geqslant -\frac{1}{2}\int_0^t \left(\sqrt{b}\omega, \varphi\right) d\tau + \left(\sqrt{b_0^{n,k}}, \varphi\right) \qquad \forall\varphi \in D(\Omega), \varphi \geqslant 0. \end{aligned} \tag{5.71}$$

5.2.4. (m,n,k)-approximation

Once again, we will approximate the initial data for the turbulent kinetic energy. We introduce $b_0^{m,n,k}$ in the following way:

$$b_0^{m,n,k} = \sum_{i=1}^{m} (b_0^{n,k}, z_i) z_i,$$

where $\{z_i\}_{i=0}^{\infty}$ denotes the orthogonal basis of $W^{1,2}(\Omega)$ and orthonormal in $L^2(\Omega)$. Using this, we consider the following problem:

$$v_{,t} + \operatorname{div}(G_k\left(|v|^2\right) v \otimes v) - \operatorname{div}\left(T_k\left(\mu^{n,m}\right) D(v)\right) = -\nabla p, \tag{5.72}$$

$$\omega_{,t} + \operatorname{div}(\omega v) - \operatorname{div}\left(T_n\left(\mu^{n,m}\right) \nabla\omega\right) = -\kappa_2\omega^2, \tag{5.73}$$

$$b_{,t} + \operatorname{div}(bv) - \operatorname{div}\left(T_n\left(\mu^{n,m}\right) \nabla b\right) = -b_+\omega + \frac{T_k(\mu^{n,m})|D(v)|^2}{1 + n^{-1}|D(v)|^2}, \tag{5.74}$$

$$\operatorname{div} v = 0 \tag{5.75}$$

in Ω^T, where $\mu^{n,m} = \frac{b_+}{\omega + \frac{1}{m}} + \frac{1}{n}$. Additionally, the problem is equipped with the periodic boundary condition and the following initial condition:

$$v_{|t=0} = v_0^n, \quad \omega_{|t=0} = \omega_0, \quad b_{|t=0} = b_0^{m,n,k}(x). \tag{5.76}$$

The following theorem states the existence result for this system:

Theorem 5.2.10. *Let us fix $k \in \mathbb{N}_+$, $n \in \mathbb{N}_+$ and $m \in \mathbb{N}_+$ such that $m \geqslant \omega_{\max}$. Then there exists a triple (c, d, ω) such that*

$$c \in W^{1,\infty}(0,T)^n, \tag{5.77}$$

$$d \in W^{1,\infty}(0,T)^m, \tag{5.78}$$

$$\omega \in L^2\left(0,T,W^{1,2}(\Omega)\right) \cap L^\infty\left(0,T,L^\infty(\Omega)\right), \tag{5.79}$$

$$\partial_t \omega \in L^2\left(0,T,W^{-1,2}(\Omega)\right), \tag{5.80}$$

$$\frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t} \leqslant \omega \leqslant \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t} \quad \text{almost everywhere in } \Omega^T, \tag{5.81}$$

which solves the problem (5.72)-(5.76) in the following sense:

$$(v_{,t}, w_i) + \left(T_k\left(\mu^{n,m}\right)D(v), D(w_i)\right) = \left(G_k\left(|v|^2\right)v\otimes v, \nabla w_i\right) \quad \text{for all } i=1,\dots,n, \tag{5.82}$$

$$(\partial_t b, z_i) - (bv, \nabla z_i) + (T_n(\mu^{n,m})\nabla b, \nabla z_i) + (b_+\omega, z_i) = \left(\frac{T_k\left(\mu^{n,m}\right)|D(v)|^2}{1+n^{-1}|D(v)|^2}, z_i\right) \quad \text{for all } i=1,\dots,m, \tag{5.83}$$

$$\langle \omega_{,t}, z\rangle - (\omega v, \nabla z) + (T_n(\mu^{n,m})\nabla\omega, \nabla z) + \kappa_2\left(\omega^2, z\right) = 0 \quad \forall z \in W^{1,2}(\Omega) \text{ a.a. } t\in(0,T), \tag{5.84}$$

where

$$\mu^{n,m} = \frac{b_+}{\omega + \frac{1}{m}} + \frac{1}{n}, \tag{5.85}$$

$$v(t,x) := \sum_{i=1}^{n} c_i(t) w_i(x), \tag{5.86}$$

$$b(t,x) := \sum_{i=1}^{m} d_i(t) z_i(x). \tag{5.87}$$

The initial data are attained strongly in the following sense:

$$\lim_{t\to 0^+} \|v(t)-v_0^n\|_2 + \|\omega(t)-\omega_0\|_2 + \left\|b(t)-b_0^{m,n,k}\right\|_2 = 0. \tag{5.88}$$

5.2.5. Proof of Theorem 5.2.10

Proof of Theorem 5.2.10. Let us recall that $\{z_i\}_{i=1}^{\infty}$ and $\{w_i\}_{i=1}^{\infty}$ denote bases of $W^{1,2}(\Omega)$ and $W^{1,2}_{\text{div}}(\Omega)$, which are orthogonal in $L^2(\Omega)$ and $L^2_{\text{div}}(\Omega)$, respectively. The proof relies on the Galerkin approximation method. We look for $\left(v^l, \omega^l, b^l\right)$ given as

$$v^l(t,x) = \sum_{i=1}^{n} c_i^l(t) w_i(x), \tag{5.89}$$

$$b^l(t,x) = \sum_{i=1}^{m} d_i^l(t) z_i(x), \tag{5.90}$$

$$\omega^l(t,x) = \sum_{i=1}^{l} e_i^l(t) z_i(x) \tag{5.91}$$

and we require that coefficients $c^l = (c_1^l, \ldots, c_n^l)$, $d^l = (d_1^l, \ldots, d_m^l)$, $e^m = (e_1^l, \ldots, e_l^l)$ solve the following system of ordinary differential equations on $(0,T)$:

$$\begin{aligned} &\left(\partial_t v^l, w_i\right) - \left(G_k\left(\left|v^l\right|^2\right) v^l \otimes v^l, \nabla w_i\right) + \left(T_k\left(\mu^l\right) D(v^l), D(w_i)\right) = 0 \\ &\qquad\qquad\qquad\qquad \text{for all } i = 1, \ldots, n, \end{aligned} \tag{5.92}$$

$$\begin{aligned} &\left(\partial_t b^l, z_i\right) - \left(b^l v^l, \nabla z_i\right) + \left(T_n(\mu^l)\nabla b^l, \nabla z_i\right) + \left(b_+^l T_m(\omega_+^l) - \frac{T_k\left(\mu^l\right)|Dv^l|^2}{1+n^{-1}|Dv^l|^2}, z_i\right) = 0 \\ &\qquad\qquad\qquad\qquad \text{for all } i = 1, \ldots, m, \end{aligned} \tag{5.93}$$

$$\begin{aligned} &\left(\partial_t \omega^l, z_i\right) - \left(\omega^l v^l, \nabla z_i\right) + \left(T_n(\mu^l)\nabla \omega^l, \nabla z_i\right) + \kappa_2\left(T_m(\omega^l)\omega_+^l, z_i\right) = 0 \\ &\qquad\qquad\qquad\qquad \text{for all } i = 1, \ldots, l, \end{aligned} \tag{5.94}$$

where

$$\mu^l = \frac{b_+^l}{\omega_+^l + \frac{1}{m}} + \frac{1}{n}. \tag{5.95}$$

We set the initial conditions for (c^l, d^l, e^l) given by

$$v^l(0) = \sum_{i=1}^{n} (v_0^n, w_i) w_i, \qquad b^l(0) = \sum_{i=1}^{m} (b_0^{m,n,k}, z_i) z_i, \qquad \omega^l(0) = \sum_{i=1}^{l} (\omega_0, z_i) z_i. \tag{5.96}$$

The existence of a solution to (5.92)-(5.96) on some small time interval follows from Carathodory's theorem. Using estimates which are established below, a solution can be extended to a time interval $[0, T]$.

l-independent estimates

Multiplying the equation (5.92) by $c_i^l(t)$ and summing from $i = 1$ through l, we get

$$\frac{1}{2}\frac{d}{dt}\|v^l\|_2^2 - \left(G_k\left(\left|v^l\right|^2\right) v^l \otimes v^l, \nabla v^l\right) + \left(T_k\left(\mu^l\right) D(v^l), D(v^l)\right) = 0. \tag{5.97}$$

We see that

$$\begin{aligned}\left(G_k\left(\left|v^l\right|^2\right) v^l \otimes v^l, \nabla v^l\right) &= \frac{1}{2}\left(G_k\left(\left|v^l\right|^2\right) v^l, \nabla\left|v^l\right|^2\right) = \frac{1}{2}\left(v^l, \nabla\Gamma_k\left(\left|v^l\right|^2\right)\right) \\ &= -\frac{1}{2}\left(\operatorname{div} v^l, \Gamma_k\left(\left|v^l\right|^2\right)\right) = 0.\end{aligned}$$

Thus, integrating (5.97) from 0 to T we have

$$\sup_{t\in(0,T)} \|v^l(t)\|_2^2 + \int_{\Omega^T} T_k(\mu^l)|Dv^l|^2 dxdt \leqslant C(\|v_0\|_2). \tag{5.98}$$

Using the orthonormality of the basis $\{w_i\}$ in $L^2(\Omega)$, we deduce that

$$\sup_{t\in(0,T)} |c^l(t)| \leqslant C(\|v_0\|_2). \tag{5.99}$$

From the equation (5.92) and the orthonormality of the basis, one can easily deduce

$$\begin{aligned}|\partial_t c_i^l| &\leqslant \left|\left(G_k\left(\left|v^l\right|^2\right) v^l \otimes v^l, \nabla w_i\right)\right| + \left|\left(T_k\left(\mu^l\right) D(v^l), D(w_i)\right)\right| \\ &\leqslant \frac{9}{2}\left|\left(G_k\left(\left|v^l\right|^2\right)\left|v^l\right|^2, |\nabla w_i|\right)\right| + \sum_{j=1}^{n}\left|c_j^l(t)\right|\left(T_k\left(\mu^l\right) D(w_j), D(w_i)\right).\end{aligned}$$

Now, using (5.20), (5.23) and the inequality (5.99), we get

$$|\partial_t c_i^l| \leqslant C(k)\left\|\nabla w_i\right\|_2 + C(n,k)\|\nabla w_i\|_2^2.$$

Using the fact that $\left\|\nabla w_i\right\|_2 \leqslant C(n)$, we get

$$\sup_{t\in(0,T)} |\partial_t c^l(t)| \leqslant C(n,k). \tag{5.100}$$

Now, multiplying (5.93) by $d_i^l(t)$ and summing from $i = 1$ through l, we obtain

$$\begin{aligned}\left(\partial_t b^l, b^l\right) - \left(b^l v^l, \nabla b^l\right) + \left(T_n(\mu^l)\nabla b^l, \nabla b^l\right)& \\ = \left(-b_+^l T_m(\omega_+^l) + \frac{T_k\left(\mu^l\right)|Dv^l|^2}{1+n^{-1}|Dv^l|^2}, b^l\right).&\end{aligned} \tag{5.101}$$

Using the fact that $\operatorname{div} v^l = 0$ and the definition (5.20), we have

$$\left(\partial_t b^l, b^l\right) + \frac{1}{n}\left(\nabla b^l, \nabla b^l\right) \leqslant \left(kn, |b^l|\right). \tag{5.102}$$

Thus, by the Young and Grönwall inequality from (5.102) we deduce

$$\sup_{t\in(0,T)} \|b^l(t)\|_2^2 + \int_0^T \|\nabla b^l\|_2^2 dt \leqslant C(k,n). \tag{5.103}$$

Using (5.103) and the orthonormality of the basis in $L^2(\Omega)$, we deduce that

$$\sup_{t\in(0,T)} \left|d^l\right| \leqslant C(k,n). \tag{5.104}$$

Now, by equation (5.93) and the orthonormality of the basis in $L^2(\Omega)$, we have

$$\begin{aligned}\left|\partial_t d_i^l\right| &\leqslant \left|\left(b^l v^l, \nabla z_i\right)\right| + \left|\left(T_n(\mu^l)\nabla b^l, \nabla z_i\right)\right| + \left|\left(-b_+^l T_m(\omega_+^l), z_i\right)\right| \\ &\quad + \left|\left(\frac{T_k\left(\mu^l\right)|Dv^l|^2}{1+n^{-1}|Dv^l|^2}, z_i\right)\right| \\ &\leqslant \left\|b^l\right\|_2 \left\|v^l\right\|_2 \|\nabla z_i\|_\infty + n\left\|\nabla b^l\right\|_2 \|\nabla z_i\|_2 + m\left\|b^l\right\|_2 \|z_i\|_2 \\ &\quad + \left\|T_k\left(\mu^l\right)|Dv^l|^2\right\|_1 \|z_i\|_\infty \\ &\leqslant \left\|b^l\right\|_2 \left\|v^l\right\|_2 \|\nabla z_i\|_\infty + n\sup_{t\in(0,T)}\left|d^l\right| \sum_{j=1}^m \|\nabla z_j\|_2 \|\nabla z_i\|_2 + m\left\|b^l\right\|_2 \|z_i\|_2 \\ &\quad + k\sup_{t\in(0,T)}\left|c^l\right|^2 \sum_{j=1}^n \|\nabla w_j\|_2^2 \|z_i\|_\infty .\end{aligned}$$

Thus, using (5.104), (5.103), (5.99), (5.98) yields

$$\sup_{t\in(0,T)} \left|\partial_t d^l\right| \leqslant C(k,n,m). \tag{5.105}$$

Now, multiplying the equation (5.94) by $e_i^l(t)$ and summing from $i = 1$ through l, we get

$$\frac{1}{2}\frac{d}{dt}\|\omega^l\|_2^2 + \left(\omega^l v^l, \nabla\omega^l\right) + \left(T_n(\mu^l)\nabla\omega^l, \nabla\omega^l\right) = -\kappa_2\left(T_m(\omega^l)\omega_+^l, \omega^l\right). \tag{5.106}$$

From $\operatorname{div} v^l = 0$ and the fact that the right-hand side is non-negative it follows

$$\frac{1}{2}\frac{d}{dt}\|\omega^l\|_2^2 + \frac{1}{n}\left(\nabla\omega^l, \nabla\omega^l\right) \leqslant 0.$$

Thus, integrating from 0 to T gives

$$\sup_{t\in(0,T)}\|\omega^l(t)\|_2^2 + \int_{\Omega^T}\|\nabla\omega^l\|_2^2 dxdt \leqslant C(n). \tag{5.107}$$

Now, using (5.107) and the equation (5.94), we can deduce that

$$\int_0^T \left\|\partial_t\omega^l\right\|_{W^{-1,2}}^2 dt \leqslant C(n,m). \tag{5.108}$$

Indeed, let us consider $\varphi \in W^{1,2}(\Omega)$ such that $\|\varphi\|_{1,2} = 1$. We can write φ in the following way:

$$\varphi = \sum_{i=1}^{l}\theta_i z_i + \overline{\varphi},$$

where $(\overline{\varphi}, z_i) = 0$ holds for $i = 1, \ldots, l$.

$$\begin{aligned}
\left|(\partial_t\omega^l, \varphi)\right| &= \left|\left(\partial_t\omega^l, \sum_{i=1}^{l}\theta_i z_i\right)\right| \\
&\leqslant \left|\left(\omega^l v^l, \nabla\sum_{i=1}^{l}\theta_i z_i\right)\right| + \left|\left(T_n(\mu^l)\nabla\omega^l, \nabla\sum_{i=1}^{l}\theta_i z_i\right)\right| \\
&\quad + \kappa_2\left|\left(T_m(\omega^l)\omega_+^l, \sum_{i=1}^{l}\theta_i z_i\right)\right| \\
&\leqslant \left\|\omega^l\right\|_2\left\|v^l\right\|_\infty\left\|\nabla\sum_{i=1}^{l}\theta_i z_i\right\|_2 + n\left\|\nabla\omega^l\right\|_2\left\|\nabla\sum_{i=1}^{l}\theta_i z_i\right\|_2 \\
&\quad + m\kappa_2\left\|\omega^l\right\|_2\left\|\sum_{i=1}^{l}\theta_i z_i\right\|_2.
\end{aligned}$$

Due to $\left\|\sum_{i=1}^{l}\theta_i z_i\right\|_2 \leqslant \|\varphi\|_2$ and $\left\|\nabla\sum_{i=1}^{l}\theta_i z_i\right\|_2 \leqslant \|\varphi\|_{1,2}$ (which both hold thanks to the Bessel inequality and $\|\sum_{i=1}^{l}\theta_i z_i\|_2^2 + \|\nabla\sum_{i=1}^{l}\theta_i z_i\|_2^2 = \left\|\sum_{i=1}^{l}\theta_i z_i\right\|_{1,2}^2 \leqslant \|\varphi\|_{1,2}^2$), the Young inequality and (5.99) we get

$$\left|(\partial_t\omega^l,\varphi)\right|^2 \leqslant C(n)\left\|\omega^l\right\|_2^2\|\varphi\|_{1,2}^2 + C(n)\left\|\nabla\omega^l\right\|_2^2\|\varphi\|_{1,2}^2 + C(m)\left\|\omega^l\right\|_2^2\|\varphi\|_2^2.$$

Thus, we get

$$\left\|\partial_t\omega^l\right\|_{-1,2}^2 = \sup_{\varphi\in W^{1,2}(\Omega),\|\varphi\|_{1,2}=1}\left|(\partial_t\omega^l,\varphi)\right|^2 \leqslant C(n,m)\left\|\omega^l\right\|_2^2 + C(n)\left\|\nabla\omega^l\right\|_2^2.$$

Finally, using (5.107) we deduce (5.108).

Taking the limit $l\to\infty$

Using estimates (5.99), (5.100) and (5.104), (5.105), we can find a subsequence (which we do not relabel) such that

$$c^l \rightharpoonup^* c \qquad \text{weakly* in } W^{1,\infty}(0,T)^n,$$

$$d^l \rightharpoonup^* d \qquad \text{weakly* in } W^{1,\infty}(0,T)^m.$$

Using the Arzela-Ascoli theorem and estimates (5.100) and (5.105), we conclude that

$$c^l \to c \qquad \text{strongly in } C(0,T)^n,$$

$$d^l \to d \qquad \text{strongly in } C(0,T)^m.$$

Based on definitions (5.89), (5.90) and the above convergences we deduce the existence of a sequence such that

$$v^l \to v = \sum_{i=1}^{n} c_i w_i \qquad \text{strongly in } C(0,T,W^{1,2}_{\mathrm{div}}(\Omega)), \tag{5.109}$$

$$b^l \to b = \sum_{i=1}^{m} d_i z_i \qquad \text{strongly in } C(0,T,W^{1,2}(\Omega)).$$

Using (5.107) and (5.108) and the Aubin-Lions lemma we obtain

$$\omega^l \rightharpoonup^* \omega \qquad \text{weakly* in } L^2\left(0,T;W^{1,2}\right) \cap L^\infty\left(0,T;L^2(\Omega)\right), \tag{5.110}$$

$$\partial_t \omega^l \rightharpoonup \partial_t \omega \qquad \text{weakly in } L^2\left(0,T;W^{-1,2}(\Omega)\right), \tag{5.111}$$

$$\omega^l \to \omega \qquad \text{strongly in } L^2\left(0,T;L^2(\Omega)\right). \tag{5.112}$$

Having the above estimates, it is easy to identify the limit of the system (5.89)-(5.96) to get

$$(\partial_t v, w_i) + \left(T_k\left(\widetilde{\mu}^{n,m}\right) D(v), D(w_i)\right) = \left(G_k\left(|v|^2\right) v \otimes v, \nabla w_i\right) \quad \text{for all } i = 1, \dots, n, \tag{5.113}$$

$$(\partial_t b, z_i) - (bv, \nabla z_i) + (T_n(\widetilde{\mu}^{n,m}) \nabla b, \nabla z_i) = \left(-b_+ T_m(\omega_+) + \frac{T_k\left(\widetilde{\mu}^{n,m}\right)|Dv|^2}{1 + n^{-1}|Dv|^2}, z_i\right) \quad \text{for all } i = 1, \dots, m, \tag{5.114}$$

$$\langle \partial_t \omega, z \rangle - (\omega v, \nabla z) + (T_n(\widetilde{\mu}^{n,m}) \nabla \omega, \nabla z) = -\kappa_2 \left(T_m(\omega)\omega_+, z\right) \quad \text{for all } z \in W^{1,2}(\Omega), \tag{5.115}$$

where

$$\widetilde{\mu}^{n,m} = \frac{b_+}{\omega_+ + \frac{1}{m}} + \frac{1}{n}.$$

To obtain the system (5.82)-(5.84), we show the bounds for ω. This will allow us to replace $\widetilde{\mu}^{n,m}$ with $\mu^{n,m}$ in equations (5.113)-(5.115). Additionally, we will be able to replace $T_m(\omega_+)$ and $T_m(\omega)$ with ω.

Minimum and maximum principle for ω

We apply ω_- as a test function in (5.115) and obtain

$$\langle \partial_t \omega, \omega_- \rangle - (\omega v, \nabla \omega_-) + \left(T_n\left(\widetilde{\mu}^{n,m}\right) \nabla \omega, \nabla \omega_-\right) = -\kappa_2 \left(T_m(\omega)\omega_+, \omega_-\right).$$

We see that the right-hand side of the above equality is equal to zero and thus

$$\frac{1}{2}\frac{d}{dt}\|\omega_-\|_2^2 - (\omega_- v, \nabla \omega_-) + \left(T_n\left(\widetilde{\mu}^{n,m}\right) \nabla \omega_-, \nabla \omega_-\right) = 0.$$

Using the fact that $\operatorname{div} v = 0$ we conclude that the second term is equal to zero. Additionally, using the non-negativity of the second term, we finally get

$$\frac{d}{dt}\|\omega_-\|_2^2 \leqslant 0 \quad \text{so} \quad \forall t \in (0,T) \quad \|\omega_-(t,\cdot)\|_2^2 = 0. \tag{5.116}$$

Thus, by (5.116), we conclude that $\omega \geqslant 0$ almost everywhere in Ω^T. From this we see that $\widetilde{\mu}^{n,m} = \mu^{n,m}$. Hence we can rewrite (5.115) in the following way:

$$\begin{aligned} \langle \partial_t \omega, z\rangle - (\omega v, \nabla z) + (T_n(\mu^{n,m})\nabla\omega, \nabla z) &= -\kappa_2 \left(T_m(\omega)\omega, z\right) \\ &\qquad \text{for all } z \in W^{1,2}(\Omega), \end{aligned} \tag{5.117}$$

Similarly, from (5.113)-(5.114) we get (5.82)-(5.83). Now, we will show the upper bound on ω. Let us first test the equation (5.117) using $(\omega - \omega_{\max})_+$:

$$\begin{aligned} \left\langle \partial_t \omega, (\omega-\omega_{\max})_+\right\rangle - \left(\omega v, \nabla(\omega - \omega_{\max})_+\right) \\ + \left(T_n\left(\mu^{n,m}\right)\nabla\omega, \nabla(\omega-\omega_{\max})_+\right) \\ = -\kappa_2\left(T_m(\omega)\omega, (\omega-\omega_{\max})_+\right). \end{aligned}$$

From this and the fact that $\operatorname{div} v = 0$ we have

$$\begin{aligned} \left\langle \partial_t(\omega-\omega_{\max})_+, (\omega-\omega_{\max})_+\right\rangle + \left(\nabla(\omega-\omega_{\max})_+ v, (\omega-\omega_{\max})_+\right) \\ + \left(T_n\left(\mu^{n,m}\right)\nabla(\omega-\omega_{\max})_+, \nabla(\omega-\omega_{\max})_+\right) \\ = -\kappa_2\left(T_m(\omega)\omega, (\omega-\omega_{\max})_+\right). \end{aligned}$$

Since $\operatorname{div} v = 0$ this gives us the following inequality:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|(\omega-\omega_{\max})_+\|_2^2 \leqslant -\kappa_2\left(T_m(\omega)\omega, (\omega-\omega_{\max})_+\right) \leqslant 0 \\ \text{so} \quad \forall t \in (0,T) \quad \left\|(\omega(t,\cdot)-\omega_{\max})_+\right\|_2 = 0. \end{aligned}$$

Thus, $\omega \leqslant \omega_{\max}$ almost everywhere in Ω^T. Let us recall that $m \geqslant \omega_{\max}$ and thus from (5.117) we get

$$\langle \partial_t\omega, z\rangle - (\omega v, \nabla z) + (T_n(\mu^{n,m})\nabla\omega, \nabla z) = -\kappa_2\left(\omega^2, z\right) \quad \text{for all } z \in W^{1,2}(\Omega), \tag{5.118}$$

which is exactly equal to (5.84). Now, let us test the equation (5.118) using $\left(\omega - \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}\right)_+$:

$$\left\langle \partial_t \omega, \left(\omega - \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}\right)_+ \right\rangle - \left(\omega v, \nabla\left(\omega - \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}\right)_+\right) + \left(T_n\left(\mu^{n,m}\right)\nabla\omega, \nabla\left(\omega - \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}\right)_+\right) = -\kappa_2\left(\omega^2, \left(\omega - \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}\right)_+\right).$$

Using the fact that $\nabla\omega = \nabla\left(\omega - \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}\right)$ we deduce that

$$\frac{1}{2}\frac{d}{dt}\|\left(\omega - \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}\right)_+\|_2^2 - \left(\frac{\kappa_2\omega_{\max}^2}{(1+\kappa_2\omega_{\max}t)^2}, \left(\omega - \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}\right)_+\right) \leqslant -\kappa_2\left(\omega^2, \left(\omega - \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}\right)_+\right).$$

Thus, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\left(\omega - \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}\right)_+\|_2^2 &\leqslant -\kappa_2\left(\omega^2 - \frac{\omega_{\max}^2}{(1+\kappa_2\omega_{\max}t)^2}, \left(\omega - \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}\right)_+\right)\\ &= -\kappa_2\left(\left(\omega + \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}\right), \left(\omega - \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}\right)_+^2\right)\\ &\leqslant 0.\end{aligned}$$

Thus, by integration and using the inequality $\omega_0 \leqslant \omega_{\max}$, we conclude that $\omega \leqslant \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}$ almost everywhere in Ω^T. Now, we will obtain the bound from below by testing the equation (5.118) by $\left(\omega - \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t}\right)_-$:

$$\left\langle \partial_t \omega, \left(\omega - \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t}\right)_- \right\rangle - \left(\omega v, \nabla\left(\omega - \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t}\right)_-\right) + \left(T_n\left(\mu^{n,m}\right)\nabla\omega, \nabla\left(\omega - \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t}\right)_-\right) = -\kappa_2\left(\omega^2, \left(\omega - \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t}\right)_-\right).$$

Again, we deduce that the second term is equal to zero and the third one is positive:

$$\frac{1}{2}\frac{d}{dt}\|\left(\omega-\frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t}\right)_-\|_2^2-\left(\frac{\kappa_2\omega_{\min}^2}{(1+\kappa_2\omega_{\min}t)^2},\left(\omega-\frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t}\right)_-\right)$$
$$\leqslant-\kappa_2\left(\omega^2,\left(\omega-\frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t}\right)_-\right).$$

Thus, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\left(\omega-\frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t}\right)_-\|_2^2 &\leqslant-\kappa_2\left(\omega^2-\frac{\omega_{\min}^2}{(1+\kappa_2\omega_{\min}t)^2},\left(\omega-\frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t}\right)_-\right)\\ &\leqslant-\kappa_2\left(\omega+\frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t},\left(\omega-\frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t}\right)_-^2\right)\\ &\leqslant 0.\end{aligned}$$

By integration and the fact that $\omega_0\geqslant\omega_{\min}$, we conclude that $\omega\geqslant\frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t}$ almost everywhere in Ω^T.

For now on we will refrain from showing the attainment of initial data. The methodology will be shown for a more complex case, that is, in the proof of Theorem 5.1.1. □

5.2.6. Proof of Theorem 5.2.9

We use (v^m,ω^m,b^m) to denote a solution of the system (5.72)-(5.76), whose existence was established in Theorem 5.2.10. Our goal is to let $m\to\infty$ and thus prove Theorem 5.2.9.

m-independent estimates

Repeating the procedure from (5.97)-(5.98) we deduce that

$$\sup_{t\in(0,T)}\|v^m(t)\|_2^2+\int_{\Omega^T}T_k(\mu^{n,m})|Dv^m|^2dxdt\leqslant C, \tag{5.119}$$

$$\sup_{t\in(0,T)}|c^m(t)|\leqslant C,\qquad \sup_{t\in(0,T)}|\partial_t c^m(t)|\leqslant C(n,k). \tag{5.120}$$

We multiply (5.83) by $d_i^m(t)$ and sum from $i=1$ through m to obtain

$$(\partial_t b^m,b^m)-(b^m v^m,\nabla b^m)+(T_n(\mu^{n,m})\nabla b^m,\nabla b^m)$$
$$=\left(-b_+^m\omega^m+\frac{T_k\left(\mu^{n,m}\right)|Dv^m|^2}{1+n^{-1}|Dv^m|^2},b^m\right).$$

Using (5.20), (5.85), (5.81), the fact that $\frac{n^{-1}|Dv^m|^2}{1+n^{-1}|Dv^m|^2} \leqslant 1$ and $\operatorname{div} v^m = 0$ we get

$$(\partial_t b^m, b^m) + \frac{1}{n}(\nabla b^m, \nabla b^m) \leqslant (kn, |b^m|).$$

Using the Young inequality and Grönwall lemma, we get

$$\sup_{t\in(0,T)} \|b^m(t)\|_2^2 + \frac{1}{n}\int_{\Omega^T} \|\nabla b^m\|_2^2 dt \leqslant C(n,k). \tag{5.121}$$

Using the equation (5.83) and (5.121), (5.119), (5.85), we conclude (in the same way as in (5.108)-(5.109)) that

$$\int_0^T \|\partial_t b^m\|_{W^{-1,2}(\Omega)}^2 \, dt \leqslant C(n,k). \tag{5.122}$$

Testing the equation (5.84) by ω^m, we get

$$\frac{1}{2}\frac{d}{dt}\|\omega^m\|_2^2 + (\omega^m v^m, \nabla\omega^m) + (T_n(\mu^{n,m})\nabla\omega^m, \nabla\omega^m) = -\kappa_2\left((\omega^m)^2, \omega^m\right).$$

Using the fact that $\operatorname{div} v^m = 0$ and (5.81), (5.85), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\omega^m\|_2^2 + \frac{1}{n}\|\nabla\omega^m\|_2^2 \leqslant 0,$$

and thus

$$\int_0^T \|\omega^m\|_{1,2}^2 \, dt \leqslant C(n,k). \tag{5.123}$$

Using the equation (5.84) and estimates (5.119), (5.123), one can deduce (again, in the same way as in (5.108)-(5.109)) that

$$\int_0^T \|\partial_t\omega^m\|_{W^{-1,2}(\Omega)}^2 \, dt \leqslant C(n,k). \tag{5.124}$$

Taking the limit $m \to \infty$

By the estimate (5.119) we can find a subsequence (which we do not relabel) such that

$$c^m \rightharpoonup^* c \quad \text{weakly* in } W^{1,\infty}(0,T)^n.$$

Using the Arzela-Ascoli theorem and the estimate (5.120), we conclude that

$$c^m \to c \qquad \text{strongly in } C(0,T)^n. \tag{5.125}$$

Based on the definition (5.86) and the convergence (5.125), one can deduce

$$v^m \to v = \sum_{i=1}^{n} c_i w_i \qquad \text{strongly in } C(0,T,W^{1,2}_{\text{div}}(\Omega)).$$

Using (5.123), (5.124) and (5.81) and the Aubin-Lions lemma, we find a subsequence such that

$$\begin{aligned}
\omega^m &\rightharpoonup \omega && \text{weakly in } L^2\left(0,T;W^{1,2}(\Omega)\right),\\
\partial_t\omega^l &\rightharpoonup \partial_t\omega && \text{weakly in } L^2\left(0,T;W^{-1,2}(\Omega)\right),\\
\omega^m &\rightharpoonup^* \omega && \text{weakly* in } L^\infty\left(0,T;L^\infty(\Omega)\right),\\
\omega^m &\to \omega && \text{strongly in } L^2\left(0,T;L^2(\Omega)\right).
\end{aligned}$$

Using (5.121), (5.122) and the Aubin-Lions lemma, we extract a subsequence such that

$$\begin{aligned}
b^m &\rightharpoonup^* b && \text{weakly* in } L^2\left(0,T;W^{1,2}(\Omega)\right) \cap L^\infty\left(0,T;L^2(\Omega)\right),\\
\partial_t b^l &\rightharpoonup \partial_t b && \text{weakly in } L^2\left(0,T;W^{-1,2}(\Omega)\right),\\
b^m &\to b && \text{strongly in } L^2\left(0,T;L^2(\Omega)\right).
\end{aligned}$$

Having the above estimates, it is easy to identify the limit of the system (5.82) - (5.86) to obtain

$$\begin{aligned}
(v_{,t}, w_i) - \left(G_k\left(|v|^2\right) v\otimes v, \nabla w_i\right) + \left(T_k\left(\widetilde{\mu^n}\right) D(v), D(w_i)\right) = 0&\\
\text{for all } i = 1,\ldots,n,&
\end{aligned} \tag{5.126}$$

$$\left\langle b_{,t}, z\right\rangle - (bv, \nabla z) + \left(T_n(\widetilde{\mu^n})\nabla b, \nabla z\right) + \left(b_+\omega - \frac{T_k\left(\widetilde{\mu^n}\right)|D(v)|^2}{1+n^{-1}|D(v)|^2}, z\right) = 0 \qquad (5.127)$$

$$\forall z \in W^{1,2}(\Omega)\text{a.a. } t \in (0,T),$$

$$\left\langle \omega_{,t}, z\right\rangle - (\omega v, \nabla z) + \left(T_n(\widetilde{\mu^n})\nabla \omega, \nabla z\right) + \kappa_2\left(\omega^2, z\right) = 0 \qquad (5.128)$$

$$\forall z \in W^{1,2}(\Omega)\text{a.a. } t \in (0,T),$$

where

$$\widetilde{\mu^n} = \frac{b_+}{\omega} + \frac{1}{n}. \qquad (5.129)$$

Now, we will show bounds for b from which we will conclude that $\widetilde{\mu^n} = \mu^n$. By doing so, we will show the existence of a solution to (5.66) - (5.69).

Minimum principle for b

Firstly, let us test the equation (5.127) with $z = b_-$. We get

$$\left\langle b_{,t}, b_-\right\rangle - (bv, \nabla b_-) + \left(T_n(\widetilde{\mu^n})\nabla b, \nabla b_-\right) = (-b_+\omega, b_-) + \left(\frac{T_k\left(\widetilde{\mu^n}\right)|D(v)|^2}{1+n^{-1}|D(v)|^2}, b_-\right).$$

Using the fact that $\operatorname{div} v = 0$ to the second term of the left-hand side, the positivity of the third term on the left-hand side and the non-positivity of the second term on the right-hand side, we get

$$\frac{1}{2}\frac{d}{dt}\|b_-\|_2^2 \leqslant (-b_+\omega+, b_-) = 0.$$

Thus we deduce that $b \geqslant 0$ almost everywhere in Ω^T. From this it follows that $\widetilde{\mu^n} = \mu^n$ and that the positive part of b in (5.127) can be dropped. Thus, the existence of a solution to the system (5.66) - (5.69) is established. Now, let us test the equation (5.67) with $z = \left(b - \frac{1}{k(1+\kappa_2\omega_{\min}t)^{\frac{1}{\kappa_2}}}\right)_-$. Again, using the equation $\operatorname{div} v = 0$ and the positivity of the third term on the left-hand side and the negativity of the second term on the right-hand side we get

$$\left\langle b_{,t}, \left(b - \frac{b_{\min}}{(1+\kappa_2\omega_{\max}t)^{\frac{1}{\kappa_2}}}\right)_-\right\rangle \leqslant \left(-b\omega, \left(b - \frac{b_{\min}}{(1+\kappa_2\omega_{\max}t)^{\frac{1}{\kappa_2}}}\right)_-\right).$$

The inequality can be rewritten in the following way:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\| \left(b - \frac{1}{k\left(1+\kappa_2\omega_{\min}t\right)^{\frac{1}{\kappa_2}}}\right)_- \right\|_2^2 \\
&\leqslant \left(-b\omega + \frac{\omega_{\max}}{k\left(1+\kappa_2\omega_{\max}t\right)^{\frac{1}{\kappa_2}+1}}, \left(b - \frac{1}{k\left(1+\kappa_2\omega_{\min}t\right)^{\frac{1}{\kappa_2}}}\right)_-\right). \\
&\leqslant \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}\left(-b + \frac{1}{k\left(1+\kappa_2\omega_{\max}t\right)^{\frac{1}{\kappa_2}}}, \left(b - \frac{1}{k\left(1+\kappa_2\omega_{\min}t\right)^{\frac{1}{\kappa_2}}}\right)_-\right) \\
&\leqslant 0.
\end{aligned}$$

Using integration from 0 to t and the fact that $b_0^{n,k} \geqslant \frac{1}{k}$, we conclude that $b \geqslant \frac{1}{k(1+\kappa_2\omega_{\min}t)^{\frac{1}{\kappa_2}}}$ almost everywhere in Ω^T, and thus we prove (5.65).

A remaining inequality

Now, we will establish (5.71). Let us test the equation (5.67) with $\frac{\varphi}{2\sqrt{b}}$, where $\varphi \in \mathcal{D}(\Omega)$ and $\varphi \geqslant 0$. Thus, we get

$$\begin{aligned}
&\langle b_{,t}, \frac{\varphi}{2\sqrt{b}}\rangle - \left(bv, \nabla\frac{\varphi}{2\sqrt{b}}\right) + \left(T_n(\mu^n)\nabla b, \nabla\frac{\varphi}{2\sqrt{b}}\right) \\
&\qquad = \left(-b\omega, \frac{\varphi}{2\sqrt{b}}\right) + \left(\frac{T_k\left(\mu^n\right)|D(v)|^2}{1+n^{-1}|D(v)|^2}, \frac{\varphi}{2\sqrt{b}}\right).
\end{aligned}$$

Using the non-negativity of the last term of the right-hand side and the equation $\operatorname{div} v = 0$, we get

$$\begin{aligned}
&\left\langle \left(\sqrt{b}\right)_t, \varphi\right\rangle + \left(\nabla bv, \frac{\varphi}{2\sqrt{b}}\right) + \left(T_n(\mu^n)\nabla b, \frac{\nabla\varphi}{2\sqrt{b}}\right) - \left(T_n(\mu^n)\nabla b, \frac{\varphi}{4b^{\frac{3}{2}}}\nabla b\right) \\
&\qquad \geqslant -\frac{1}{2}\left(\sqrt{b}\omega, \varphi\right).
\end{aligned}$$

Let us observe that the last term of the left-hand side is not positive. Thus, we obtain

$$\left\langle \left(\sqrt{b}\right)_t, \varphi\right\rangle + \left(\nabla\left(\sqrt{b}\right)v, \varphi\right) + \left(T_n(\mu^n)\nabla\sqrt{b}, \nabla\varphi\right) \geqslant -\frac{1}{2}\left(\sqrt{b}\omega, \varphi\right).$$

After integrating from 0 to t we get

$$\left(\sqrt{b}(t),\varphi\right)-\int_0^t\left(\sqrt{b}v,\nabla\varphi\right)d\tau+\int_0^t\left(T_n(\mu^n)\nabla\sqrt{b},\nabla\varphi\right)d\tau \geqslant\left(\sqrt{b_0^{n,k}},\varphi\right)-\frac{1}{2}\int_0^t\left(\sqrt{b}\omega,\varphi\right)d\tau.$$

5.2.7. Proof of Theorem 5.2.8

Let (v^n,b^n,ω^n) be a solution to the problem (5.66)-(5.69), whose existence is guaranteed by Theorem 5.2.9. Our goal is to pass with $n\to\infty$ in (5.66)-(5.69) to obtain (5.45)-(5.47), and thus to prove Theorem 5.2.8.

n-independent estimates

Proceeding as before, we get

$$\|v^n(t)\|_2^2+2\int_0^t\int_\Omega T_k(\mu^n)|D(v^n)|^2dxd\tau=\|v_0^n\|_2^2. \tag{5.130}$$

Thus, we deduce the following estimate:

$$\sup_{t\in(0,T)}\|v^n(t)\|_2^2+\int_{\Omega^T}T_k(\mu^n)|D(v^n)|^2dxd\tau\leqslant C. \tag{5.131}$$

Now, using bounds on ω (5.64) and b (5.65) and the Korn inequality, we deduce

$$\int_0^T\|v^n\|_{1,2}^2\,dxd\tau\leqslant C(k). \tag{5.132}$$

By the equation (5.66) and the inequality (5.132) one can deduce

$$\int_0^T\|\partial_t v^n\|_{-1,2}^2\,dxd\tau\leqslant C(k). \tag{5.133}$$

Using the standard interpolation inequality $\|u\|_{\frac{10}{3}}\leqslant C\,\|u\|_2^{\frac{2}{5}}\,\|u\|_{1,2}^{\frac{3}{5}}$ and (5.131), (5.132), we get

$$\int_0^T\|v^n\|_{\frac{10}{3}}^{\frac{10}{3}}\,d\tau\leqslant C(k). \tag{5.134}$$

Now, we will concentrate on uniform estimates for b^n. First, for arbitrary $a > 0$ we set $z = T_a(b^n)$ in (5.67) and obtain

$$\langle b^n_{,t}, T_a(b^n)\rangle - (b^n v^n, \nabla T_a(b^n)) + (T_n(\mu^n)\nabla b^n, \nabla T_a(b^n)) \\ = \left(-b^n\omega^n + \frac{T_k(\mu^n)\,|Dv^n|^2}{1+n^{-1}|Dv^n|^2}, T_a(b^n)\right).$$

Using the definition (5.21) we get

$$\langle \partial_t b^n, T_a(b^n)\rangle = \frac{d}{dt}\left\|\Theta_a(b^n)\right\|_1. \tag{5.135}$$

From the inequality (5.131) it follows that

$$\int_0^T \left(\frac{T_k(\mu^n)\,|Dv^n|^2}{1+n^{-1}|Dv^n|^2}, T_a(b^n)\right) \leqslant aC. \tag{5.136}$$

By the fact that $\operatorname{div} v^n = 0$ we get

$$(b^n v^n, \nabla T_a(b^n)) = -\left(\nabla b^n v^n, T_a(b^n)\right) = -\left(v^n, \nabla\Theta_a(b^n)\right) = \left(\operatorname{div} v^n, \Theta_a(b^n)\right) = 0.$$

We also have

$$(T_n(\mu^n)\nabla b^n, \nabla T_a(b^n)) = \int_\Omega T_n(\mu^n)\left|\nabla T_a(b^n)\right|^2 dx. \tag{5.137}$$

Combining (5.135), (5.136) and (5.137) we obtain

$$\frac{d}{dt}\left\|\Theta_a(b^n)\right\|_1 + \int_\Omega T_n(\mu^n)\left|\nabla T_a(b^n)\right|^2 dx \leqslant Ca.$$

By integration of the above inequality from 0 to T we get

$$\sup_{t\in(0,T)}\left\|\Theta_a(b^n(t))\right\|_1 + \int_{\Omega^T} T_n(\mu^n)\left|\nabla T_a(b^n)\right|^2 dx d\tau \leqslant CaT + 2\left\|\Theta_a(b^n(0))\right\|_1. \tag{5.138}$$

Using (5.21) one can show that

$$\text{if}\ \ b^n \geqslant a \quad \text{then} \quad \Theta_a(b^n) \geqslant \frac{1}{2}ab^n$$

and

$$\text{if } \ b^n < a \quad \text{then} \quad \Theta_a(b^n) = \frac{1}{2}(b^n)^2.$$

Thus, we get

$$\begin{aligned} \|b^n(t)\|_1 &\leqslant \int_{b^n(t)\geqslant a} \frac{2}{a}\Theta_a(b^n(t))dx + \int_{b^n(t)<a} \sqrt{2\Theta_a(b^n(t))}dx \\ &\leqslant C(a)\|\Theta_a(b^n(t))\|_1 + C(a). \end{aligned} \tag{5.139}$$

Using (5.138), (5.139) and (5.51) the following inequality follows:

$$\sup_{t\in(0,T)} \|b^n(t)\|_1 + \int_{\Omega^T} T_n(\mu^n)\left|\nabla T_a(b^n)\right|^2 dxd\tau \leqslant C(a). \tag{5.140}$$

Now, we test the equation (5.67) inserting $z = \frac{1}{(b^n)^\lambda}$ (which is a viable test function based on (5.65)), where $\lambda \in (0,1)$:

$$\frac{d}{dt}\int_\Omega \frac{(b^n)^{1-\lambda}}{1-\lambda}dx - \lambda\int_\Omega T_n(\mu^n)\frac{|\nabla b^n|^2}{(b^n)^{1+\lambda}}dx = \left(-b^n\omega^n + \frac{T_k(\mu)\,|D(v)|^2}{1+n^{-1}|D(v)|^2}, \frac{1}{(b^n)^\lambda}\right).$$

Thus, integrating from 0 to t and taking supremum over $t \in (0,T)$, we obtain the following inequality:

$$\lambda\int_{\Omega^T} T_n(\mu^n)\frac{|\nabla b^n|^2}{(b^n)^{1+\lambda}}dxdt \leqslant \int_0^T \left(b^n\omega^n, \frac{1}{(b^n)^\lambda}\right)dt + \frac{1}{1-\lambda}\sup_{t\in(0,T)}\int_\Omega (b^n)^{1-\lambda}\,dx.$$

Using (5.140) and (5.64), we can bound the right-hand side uniformly and finally obtain

$$\int_{\Omega^T} \frac{T_n(\mu^n)}{(b^n)^{1+\lambda}}|\nabla b^n|^2 dxdt \leqslant C(\lambda^{-1}) \quad \forall \ \lambda \in (0,1). \tag{5.141}$$

Now, we set $z = \frac{1}{b^n}$ in (5.67), and using the fact that $\operatorname{div} v^n = 0$ we obtain

$$\frac{d}{dt}\int_\Omega \ln b^n(t)dx - \int_\Omega \frac{T_n(\mu^n)}{(b^n)^2}|\nabla b^n|^2 dx = \left(-b^n\omega^n + \frac{T_k(\mu^n)\,|Dv^n|^2}{1+n^{-1}|Dv^n|^2}, \frac{1}{b^n}\right).$$

After integrating from 0 to t we deduce

$$-\int_{b^n<1}\ln b^n(t)dx+\int_0^t\int_\Omega \frac{T_n(\mu^n)}{(b^n)^2}|\nabla b^n|^2dxdt+\int_0^t\left(\frac{T_k(\mu^n)|Dv^n|^2}{1+n^{-1}|Dv^n|^2},\frac{1}{b^n}\right)dt$$
$$\leqslant \int_0^t(\omega^n,1)+\|\ln b_0^{n,k}\|_1.$$

With the proper usage of supremum we obtain

$$\sup_{t\in(0,T)}-\int_{b^n<1}\ln b^n(t)dx+\int_0^T\int_\Omega \frac{T_n(\mu^n)}{(b^n)^2}|\nabla b^n|^2dxdt+\int_0^T\left(\frac{T_k(\mu^n)|Dv^n|^2}{1+n^{-1}|Dv^n|^2},\frac{1}{b^n}\right)dt$$
$$\leqslant 2\int_0^T(\omega^n,1)\,dt+2\|\ln b_0^{n,k}\|_1.$$

By the fact that

$$\|\ln b^n(t)\|_1=-\int_{\Omega\cap\{b^n(t)<1\}}\ln b^n(t)dx+\int_{\Omega\cap\{b^n(t)>1\}}\ln b^n(t)dx,$$

it follows

$$\sup_{t\in(0,T)}\|\ln b^n(t)\|_1+\int_0^T\int_\Omega \frac{T_n(\mu^n)}{(b^n)^2}|\nabla b^n|^2dxdt+\int_0^T\left(\frac{T_k(\mu^n)|Dv^n|^2}{1+n^{-1}|Dv^n|^2},\frac{1}{b^n}\right)dt$$
$$\leqslant 2\int_0^T(\omega^n,1)\,dt+\sup_{t\in(0,T)}\|b^n(t)\|_1+2\|\ln b_0^{n,k}\|_1.$$

Thus, by (5.64), (5.140) (taken with $a=1$), (5.2), (5.51), (5.52) we get

$$\sup_{t\in(0,T)}\|\ln b^n(t)\|_1+\int_0^T\int_\Omega \frac{T_n(\mu^n)}{(b^n)^2}|\nabla b^n|^2dxdt+\int_0^T\left(\frac{T_k(\mu^n)|D(v)|^2}{1+n^{-1}|D(v)|^2},\frac{1}{b^n}\right)dt\leqslant C. \tag{5.142}$$

Combining (5.142) with (5.141), (5.52), (5.51) yields

$$\begin{aligned}&\sup_{t\in(0,T)}\|\ln b^n(t)\|_1+\int_0^T\int_\Omega \frac{T_n(\mu^n)}{(b^n)^{1+\lambda}}|\nabla b^n|^2dxdt+\int_0^T\int_\Omega\frac{T_k(\mu^n)|D(v)|^2}{b^n(1+n^{-1}|D(v)|^2)}dxdt\\&\qquad\leqslant C(\lambda^{-1})\quad\forall\ \ \lambda\in(0,1].\end{aligned} \tag{5.143}$$

We see that, based on (5.131), (5.20), for all $k\in\mathbb{N}$

$$k\int_{\Omega^T\cap\{\mu^n\geqslant k\}}\frac{|Dv^n|^2}{1+n^{-1}|Dv^n|^2}dxdt\leqslant C.$$

Additionally, by (5.143) with some specific λ i.e. $\lambda = \frac{1}{2}$, (5.20), (5.64), (5.57), we have that for all $k \in \mathbb{N}$

$$\int_{\Omega^T \cap \{\mu^n \leqslant k\}} \frac{|Dv^n|^2}{1 + n^{-1}|Dv^n|^2} dxdt \leqslant C.$$

Hence

$$\int_0^T \int_\Omega \frac{|Dv^n|^2}{1 + n^{-1}|Dv^n|^2} dxdt \leqslant C, \tag{5.144}$$

where C does not depend on k and n. Next, we will focus on estimates of $T_n(\mu^n)$ that are uniform with respect to n and k. Using the definition (5.20), we get

$$T_n\left(\frac{b^n}{\omega^n}\right) \leqslant T_n\left(\mu^n\right) \leqslant T_n\left(\frac{b^n}{\omega^n}\right) + \frac{1}{n}$$

and thus

$$\min\left\{1, \frac{1}{\omega^n}\right\} T_n\left(b^n\right) \leqslant T_n\left(\mu^n\right) \leqslant \max\left\{1, \frac{1}{\omega^n}\right\} T_n\left(b^n\right) + \frac{1}{n}.$$

Due to (5.64), we finally get

$$C_1 T_n\left(b^n\right) \leqslant T_n\left(\mu^n\right) \leqslant C_2 T_n\left(b^n\right) + \frac{1}{n}. \tag{5.145}$$

Thus, by (5.143)

$$\begin{aligned}
&\int_0^T \|\nabla\left[(T_n(b^n))^{1-\frac{\lambda}{2}}\right]\|_2^2 dt = C(\lambda)\int_{\Omega^T} \frac{|\nabla T_n(b^n)|^2}{(T_n(b^n))^\lambda} dxdt \\
&= C(\lambda)\int_{\Omega^T \cap \{b^n \leqslant n\}} \frac{|\nabla T_n(b^n)|^2}{(b^n)^\lambda} dxdt = C(\lambda)\int_{\Omega^T \cap \{b^n \leqslant n\}} \frac{T_n(b^n)}{(b^n)^{1+\lambda}} |\nabla T_n(b^n)|^2 dxdt \\
&\leqslant C(\lambda)\int_{\Omega^T \cap \{b^n \leqslant n\}} \frac{T_n(\mu^n)}{(b^n)^{1+\lambda}} |\nabla T_n(b^n)|^2 dxdt \leqslant C(\lambda^{-1}).
\end{aligned} \tag{5.146}$$

Combining (5.140) (i.e. with $a = 1$) with (5.146) gives

$$\sup_{t \in (0,T)} \|\left(T_n\left(b^n\right)\right)^{1-\lambda/2}\|_1 + \int_0^T \left\|\nabla\left[(T_n(b^n))^{1-\frac{\lambda}{2}}\right]\right\|_2^2 dt \leqslant C(\lambda^{-1}). \tag{5.147}$$

Applying the interpolation inequality $\|f\|_{\frac{8}{3}}^{\frac{8}{3}} \leqslant C\|f\|_1^{\frac{2}{3}} \|f\|_{1,2}^2$ and inequality (5.147) we get

$$\int_{\Omega^T} \left\|(T_n(\mu^n))^{1-\lambda/2}\right\|_{\frac{8}{3}}^{\frac{8}{3}} \leqslant C(\lambda^{-1})$$

and hence

$$\int_{\Omega^T} \|T_n(\mu^n)\|_{\frac{8-4\lambda}{3}}^{\frac{8-4\lambda}{3}} \leqslant C(\lambda^{-1}) \text{ for all } \lambda \in (0,1). \tag{5.148}$$

Moreover, let us observe that (5.148) implies

$$\int_{\Omega^T} \|(T_n(\mu^n))^{\alpha}\|_q^q \leqslant C(q) \text{ for all } q \in \left[1, \frac{8}{3\alpha}\right) \text{ and } \alpha \in (0,1]. \tag{5.149}$$

Now, we continue with k-dependent estimates that will be useful to obtain $n \to \infty$ limit. From (5.143), the maximum principle for ω^n (5.64) and minimum principle for b^n (5.65) we get

$$\int_{\Omega^T} \frac{|\nabla b^n|^2}{(b^n)^{1+\lambda}} dxdt \leqslant C(\lambda^{-1}, k),$$

which combined with (5.140) yields

$$\int_0^T \left\|(b^n)^{\frac{1-\lambda}{2}}\right\|_{1,2}^2 dxdt \leqslant C(\lambda^{-1}, k).$$

Using the embedding $W^{1,2} \subset L^6$ implies

$$\int_0^T \left\|(b^n)^{1-\lambda}\right\|_3 dxdt \leqslant C(\lambda^{-1}, k). \tag{5.150}$$

Finally, applying the interpolation inequality

$$\left\|(b^n)^{1-\lambda}\right\|_{\frac{5}{3}}^{\frac{5}{3}} \leqslant C\| (b^n)^{1-\lambda} \|_1^{\frac{2}{3}} \left\|(b^n)^{1-\lambda}\right\|_3$$

and (5.140), (5.150), we conclude that

$$\int_0^T \|b^n\|_{\frac{5-\lambda}{3}}^{\frac{5-\lambda}{3}} dxdt \leqslant C(\lambda^{-1}, k). \tag{5.151}$$

Now, we proceed with k dependent estimates on the diffusion term. For any $q \in (1, \frac{5}{4})$, due to (5.151) and (5.143) we get

$$\begin{aligned}\int_{\Omega^T} \left|\sqrt{T_n(\mu^n)}\nabla b^n\right|^q dxdt &= \int_{\Omega^T} \left(\frac{T_n(\mu^n)\,|\nabla b^n|^2}{(b^n)^{1+\lambda}}\right)^{q/2} (b^n)^{\frac{(1+\lambda)q}{2}} dxdt \\ &\leqslant \left(\int_{\Omega^T} \frac{T_n(\mu^n)\,|\nabla b^n|^2}{(b^n)^{1+\lambda}} dxdt\right)^{q/2} \left(\int_{\Omega^T} (b^n)^{\frac{(1+\lambda)q}{2-q}} dxdt\right)^{\frac{2-q}{2}} \\ &\leqslant C(\lambda^{-1}, k),\end{aligned}$$

where $\lambda < \frac{5}{3}\frac{2-q}{q} - 1$, which implies that $\frac{(1+\lambda)q}{2-q} < \frac{5}{3}$. Therefore, we have the following estimate:

$$\int_{\Omega^T} \left|\sqrt{T_n(\mu^n)}\nabla b^n\right|^{\frac{5-\lambda}{4}} dxdt \leqslant C(\lambda^{-1}, k). \tag{5.152}$$

Thus, combining the above inequality with (5.57), (5.64), (5.65) and (5.151), we obtain

$$\int_{\Omega^T} \|b^n\|_{1,\frac{5-\lambda}{4}}^{\frac{5-\lambda}{4}} dxdt \leqslant C(\lambda^{-1}, k). \tag{5.153}$$

Notice that from $\frac{79}{80-\lambda} > \frac{79}{80} = \frac{4}{5} + \frac{3}{16}$, the Hölder inequality, (5.152) and (5.148) it follows that

$$\int_{\Omega^T} |T_n(\mu^n)\nabla b^n|^{\frac{80-\lambda}{79}} dxdt \leqslant C(\lambda^{-1}, k). \tag{5.154}$$

Additionally, we can observe that due to (5.154), (5.145), (5.44),

$$\int_{\Omega^T} |(T_n(\mu^n))^{\alpha}\,\nabla b^n|^{\frac{80-\lambda}{79}} dxdt \leqslant C(\lambda^{-1}, k) \text{ for all } \alpha \in [0,1]. \tag{5.155}$$

In order to obtain a uniform bound on $\partial_t b^n$, it remains to estimate the convective term $b^n v^n$. Let us observe that

$$\|b^n v^n\|_{\frac{10-\lambda}{9}} \leqslant \|v^n\|_{\frac{10}{3}} \|b^n\|_{\frac{10}{3}\frac{10-\lambda}{20+\lambda}}, \tag{5.156}$$

where $\frac{10}{3}\frac{10-\lambda}{20+\lambda} \in (10/7, 5/3)$, and so $\frac{10}{3}\frac{10-\lambda}{20+\lambda} < \frac{5-\lambda^*}{3}$ for some $\lambda^* \in (0,1)$. Thus, we can conclude that based on (5.151), (5.134) and (5.156) we have

$$\|b^n v^n\|_{\frac{10-\lambda}{9}} \leqslant C(k, \lambda^{-1}). \tag{5.157}$$

Using the equation (5.67) tested with $z \in W^{1,\frac{80-\lambda}{1-\lambda}}(\Omega)$, combined with inequalities (5.157), (5.154), (5.64), (5.151), (5.131) and the embedding $W^{1,\frac{80-\lambda}{1-\lambda}}(\Omega) \subset W^{1,80}(\Omega) \subset C^{0,\frac{77}{80}}(\Omega)$, we conclude

$$\int_0^T \|\partial_t b^n\|_{-1,\frac{80-\lambda}{79}} \leqslant C(k, \lambda^{-1}). \tag{5.158}$$

Finally, we derive k-independent estimates for ω^n. We set testing function $z = \omega^n$ in (5.68) and obtain

$$\frac{1}{2}\frac{d}{dt}\|\omega^n\|_2^2 + \int_\Omega T_n(\mu^n)\,|\nabla \omega^n|^2 = -\kappa_2\left((\omega^n)^2, \omega^n\right).$$

Thus, integrating from 0 to T we get

$$\int_{\Omega^T} T_n(\mu^n)\,|\nabla \omega^n|^2\,dxdt \leqslant C, \tag{5.159}$$

which, after using (5.64) and (5.65), implies that

$$\int_0^T \|\omega^n\|_{1,2}^2\,dxdt \leqslant C(k). \tag{5.160}$$

We see that due to the Hölder inequality,

$$\begin{aligned} \|T_n(\mu^n)\nabla\omega^n\|_{\frac{16-\lambda}{11}} &\leqslant \left\|\sqrt{T_n(\mu^n)}\nabla\omega^n\right\|_2 \left\|\sqrt{T_n(\mu^n)}\right\|_{2\frac{16-\lambda}{6+\lambda}} \\ &\leqslant \left\|\sqrt{T_n(\mu^n)}\nabla\omega^n\right\|_2 \|T_n(\mu^n)\|^2_{\frac{16-\lambda}{6+\lambda}}. \end{aligned} \tag{5.161}$$

We see that $\frac{16-\lambda}{6+\lambda} \in (15/7, 8/3)$, and so $\frac{16-\lambda}{6+\lambda} \leqslant \frac{8-\lambda^*}{3}$ for some $\lambda^* \in (0,1)$. From this and (5.159), (5.148) we obtain

$$\|T_n(\mu^n)\nabla\omega^n\|_{\frac{16-\lambda}{11}} \leqslant C(\lambda^{-1}). \tag{5.162}$$

Thus, having (5.162), (5.134), (5.64) and (5.68), we can conclude that

$$\int_0^T \|\partial_t \omega^n\|_{-1,\frac{16-\lambda}{11}}^{\frac{16-\lambda}{11}} \, dxdt \leqslant C(\lambda^{-1}). \tag{5.163}$$

Passing to the limit with n

Having (5.131), (5.132), (5.133), (5.134), (5.153), (5.158), (5.160), (5.163), we can find a subsequence (which we do not relabel) such that

$$
\begin{aligned}
v^n &\rightharpoonup^* v && \text{weakly* in } L^\infty\left(0,T,L^2_{\text{div}}\right) \cap L^2\left(0,T,W^{1,2}_{\text{div}}(\Omega)\right), && (5.164)\\
\partial_t v^n &\rightharpoonup \partial_t v && \text{weakly in } L^2\left(0,T,W^{-1,2}_{\text{div}}(\Omega)\right), && (5.165)\\
b^n &\rightharpoonup^* b && \text{weakly* in } L^q\left(0,T,W^{1,q}(\Omega)\right) \cap L^\infty(0,T,L^1(\Omega)) \text{ for all } q \in [1,5/4), && (5.166)\\
\partial_t b^n &\rightharpoonup \partial_t b && \text{weakly in } \mathcal{M}\left(0,T,W^{-1,q}(\Omega)\right) \text{ for all } q \in [1,80/79), && (5.167)\\
\omega^n &\rightharpoonup^* \omega && \text{weakly* in } L^2\left(0,T,W^{1,2}(\Omega)\right) \cap L^\infty(0,T,L^\infty(\Omega)), && (5.168)\\
\partial_t \omega^n &\rightharpoonup \partial_t \omega && \text{weakly in } L^q\left(0,T,W^{-1,q}(\Omega)\right) \text{ for all } q \in [1,16/11). && (5.169)
\end{aligned}
$$

Now, using the Aubin-Lions Lemma 5.2.6, we conclude that for $\alpha \in (0,1)$ we have

$$
\begin{aligned}
v^n &\to v && \text{strongly in } L^2\left(0,T,W^{\alpha,2}(\Omega) \cap L^2_{\text{div}}(\Omega)\right), && (5.170)\\
\omega^n &\to \omega && \text{strongly in } L^2\left(0,T,W^{\alpha,2}(\Omega)\right), && (5.171)\\
b^n &\to b && \text{strongly in } L^q\left(0,T,W^{\alpha,q}(\Omega)\right) \quad \text{for all } q \in \left[1,\frac{5}{4}\right). && (5.172)
\end{aligned}
$$

We can extract subsequences that converge almost everywhere

$$
\begin{aligned}
v^n &\to v && \text{almost everywhere in } \Omega^T, && (5.173)\\
\omega^n &\to \omega && \text{almost everywhere in } \Omega^T, && (5.174)\\
b^n &\to b && \text{almost everywhere in } \Omega^T. && (5.175)
\end{aligned}
$$

Moreover, from (5.151), (5.173), (5.134), (5.175) and the Vitali Lemma 5.2.3, we get

$$
\begin{aligned}
v^n &\to v && \text{strongly in } L^q\left(0,T,L^q(\Omega)\right) \text{ for all } q \in [1,10/3), && (5.176)\\
b^n &\to b && \text{strongly in } L^q\left(0,T,L^q(\Omega)\right) \text{ for all } q \in [1,5/3). && (5.177)
\end{aligned}
$$

Now, we will identify the limit of (5.68) as $n \to \infty$. Notice that (5.162) implies

$$T_n(\mu^n)\nabla\omega^n \rightharpoonup \overline{\mu\nabla\omega} \quad \text{in } L^q(\Omega^T) \text{ for all } q \in [1, 16/11). \tag{5.178}$$

Thus, using (5.170), (5.171), (5.169), we get

$$\langle \omega_{,t}, z\rangle - (\omega v, \nabla z) + \left(\overline{\mu\nabla\omega}, \nabla z\right) = -\kappa_2\left(\omega^2, z\right) \qquad \forall z \in W^{1,\infty}(\Omega) \text{ a.a. } t \in (0,T).$$

We need to show that $\overline{\mu\nabla\omega} = \mu\nabla\omega$ almost everywhere in Ω^T. To do so we use Lemma 5.2.3 and (5.148), (5.174), (5.175) to obtain

$$T_n(\mu^n) \to \mu \text{ in } L^q(0,T,L^q(\Omega)) \text{ for all } q \in [1, 8/3). \tag{5.179}$$

Then, from (5.179), (5.168) and Lemma 5.2.4 we conclude

$$T_n(\mu^n)\nabla\omega^n \rightharpoonup \mu\nabla\omega \text{ in } L^q(0,T,L^q(\Omega)) \text{ for all } q \in \left[1, \frac{8}{7}\right). \tag{5.180}$$

Using (5.180), (5.178) and the uniqueness of a weak limit, we get $\overline{\mu\nabla\omega} = \mu\nabla\omega$. Now, we will focus on obtaining a weak limit in (5.67). From (5.149), (5.174), (5.175) and the Vitali Lemma 5.2.3, for all $\alpha \in (0,1)$ we have

$$(T_n(\mu^n))^\alpha \to \mu^\alpha \text{ strongly in } L^q(0,T,L^q(\Omega)) \text{ for all } q \in \left[1, \frac{8}{3\alpha}\right). \tag{5.181}$$

From (5.155) and the fact that $\frac{81}{80} < \frac{80}{79}$ we get

$$(T_n(\mu^n))^\alpha \nabla b^n \rightharpoonup \overline{\mu^\alpha\nabla b} \text{ weakly in } L^{\frac{81}{80}}\left(0,T; L^{\frac{81}{80}}(\Omega)\right) \text{ for all } \alpha \in [0,1]. \tag{5.182}$$

Our goal is to identify this limit for all $\alpha \in [0,1]$. We will proceed inductively. Define $h = \frac{1}{81}$, $\alpha_0 = 0$ and $\alpha_{i+1} = \alpha_i + h$. Notice that (5.182) holds for α_0 due to (5.166). Assume that it holds for α_i, that is

$$(T_n(\mu^n))^{\alpha_i} \nabla b^n \rightharpoonup \mu^{\alpha_i}\nabla b \text{ weakly in } L^{\frac{81}{80}}\left(0,T; L^{\frac{81}{80}}(\Omega)\right). \tag{5.183}$$

Using (5.183), (5.181) with $\alpha = h$ and $q = \frac{4\cdot 81}{3}$ and Lemma 5.2.4 we get

$$\begin{aligned}(T_n(\mu^n))^{\alpha_{i+1}} \nabla b^n &= (T_n(\mu^n))^h (T_n(\mu^n))^{\alpha_i} \nabla b^n \\ &\rightharpoonup \mu^h \mu^{\alpha_i} \nabla b = \mu^{\alpha_{i+1}} \nabla b \text{ weakly in } L^{\frac{324}{323}}\left(0,T;L^{\frac{324}{323}}(\Omega)\right).\end{aligned} \tag{5.184}$$

From (5.182), (5.184) and the uniqueness of the weak limit it follows

$$(T_n(\mu^n))^{\alpha_{i+1}} \nabla b^n \rightharpoonup \mu^{\alpha_{i+1}} \nabla b \text{ weakly in } L^{\frac{81}{80}}\left(0,T;L^{\frac{81}{80}}(\Omega)\right).$$

Thus, setting $i = 81$ we obtain

$$T_n(\mu^n)\nabla b^n \rightharpoonup \mu \nabla b \text{ weakly in } L^{\frac{81}{80}}\left(0,T;L^{\frac{81}{80}}(\Omega)\right). \tag{5.185}$$

From (5.154), we deduce that $T_n(\mu^n)\nabla b^n \rightharpoonup \overline{\mu\nabla b}$ weakly in $L^q(0,T,L^q(\Omega))$ for all $q \in [1, 80/79)$. Finally, the uniqueness of the weak limit and (5.185) imply

$$T_n(\mu^n)\nabla b \rightharpoonup \mu \nabla b \text{ weakly in } L^q(0,T,L^q(\Omega)) \text{ for all } q \in [1, 80/79). \tag{5.186}$$

From (5.174), (5.175), (5.20) and Lemma 5.2.3, we can conclude that for all $\alpha \in (0,1)$

$$(T_k(\mu^n))^\alpha \rightharpoonup (T_k(\mu))^\alpha \text{ strongly in } L^q(0,T,L^q(\Omega)) \text{ for all } q \in [1,\infty). \tag{5.187}$$

From (5.131), (5.164), (5.187) with $\alpha = \frac{1}{2}$, Lemma 5.2.4 and the uniqueness of the weak limit, we have

$$\sqrt{T_k(\mu^n)} Dv^n \rightharpoonup \sqrt{T_k(\mu)} Dv \text{ weakly in } L^2\left(0,T;L^2(\Omega)\right). \tag{5.188}$$

Now, using (5.188), (5.187) with $\alpha = \frac{1}{2}$, (5.165), (5.134), (5.170), we can pass to the limit in (5.66) to get

$$\begin{aligned}\langle v_{,t}, w\rangle - \left(G_k\left(|v|^2\right) v \otimes v, \nabla w\right) + \left(T_k(\mu) D(v), D(w)\right) &= 0 \\ \forall w \in W^{1,2}_{\text{div}}(\Omega) \text{ a.a. } t \in (0,T).&\end{aligned} \tag{5.189}$$

The solution is defined on time interval $[0, T)$, however by repeating all previous steps it can also be attained on time interval $[0, T + \varepsilon)$. We will consider such an extended solution to obtain stronger convergence results

$$\langle v_{,t}, w\rangle - \left(G_k\left(|v|^2\right) v \otimes v, \nabla w\right) + \left(T_k(\mu) D(v), D(w)\right) = 0 \qquad \forall w \in W^{1,2}_{\mathrm{div}}(\Omega) \text{ a.a. } t \in (0, T + \varepsilon). \tag{5.190}$$

First, notice that based on (5.170) and Lemma 5.2.7 we have

$$v^n(t) \to v(t) \text{ in } L^2(\Omega) \text{ for almost all } t \in (0, T + \varepsilon). \tag{5.191}$$

Let us pick a time $t^* \in (T, T + \varepsilon)$ such that convergence (5.191) holds. Now, let us set $w = v$ (which is a viable test function) in (5.190) and integrate from 0 to t^*

$$\|v(t^*)\|_2^2 + 2\int_{\Omega^{t^*}} T_k(\mu)|D(v)|^2 dxdt = \|v_0\|_2^2.$$

By setting $t = t^*$ in (5.130) (having in mind that it is valid for the extended solution) and passing with $n \to \infty$, we get

$$\limsup_{n\to\infty} \left(\|v^n(t^*)\|_2^2 + 2\int_{\Omega^{t^*}} T_k(\mu^n)|Dv^n|^2 dxdt\right) = \|v_0\|_2^2.$$

By (5.191) we get

$$\|v(t^*)\|_2^2 + 2\limsup_{n\to\infty}\int_{\Omega^{t^*}} T_k(\mu^n)|Dv^n|^2 dxdt = \|v_0\|_2^2. \tag{5.192}$$

By subtraction it follows

$$\limsup_{n\to\infty}\int_{\Omega^{t^*}} T_k(\mu^n)|Dv^n| dxdt = \int_{\Omega^{t^*}} T_k(\mu)|D(v)|^2 dxdt. \tag{5.193}$$

Thus, using (5.193) and (5.188) (again, having in mind that it can be attained up to time $T + \varepsilon$), we conclude that

$$\sqrt{T_k(\mu^n)}Dv^n \to \sqrt{T_k(\mu)}Dv \text{ strongly in } L^2\left(0, t^*; L^2(\Omega)\right)$$

and thus, due to the fact that $t^* \in (T, T+\varepsilon)$,

$$T_k(\mu^n)|Dv^n|^2 \to T_k(\mu)|Dv|^2 \text{ strongly in } L^1\left(0,T;L^1(\Omega)\right). \tag{5.194}$$

Having this, we can extract a subsequence that converges almost everywhere

$$T_k(\mu^n)|Dv^n|^2 \to T_k(\mu)|Dv|^2 \text{ almost everywhere in } \Omega^T. \tag{5.195}$$

From (5.144) we have

$$\int_0^T \int_\Omega \frac{T_k(\mu^n)|Dv^n|^2}{T_k(\mu^n) + n^{-1}T_k(\mu^n)|Dv^n|^2} dxdt \leqslant C.$$

Using (5.195), (5.174), (5.175) and the Fatou lemma yields

$$\int_{\Omega^T} \frac{T_k(\mu)|Dv|^2}{T_k(\mu)} dxdt = \int_{\Omega^T} |Dv|^2 dxdt \leqslant C.$$

Now, we can strengthen (5.167). We see that, based on (5.67) and (5.194), (5.176), (5.177), (5.168), (5.186), we have

$$\begin{aligned}
\int_0^T \langle \partial_t b^n, z\rangle dt = &\int_0^T (b^n v^n, \nabla z)\, dt - \int_0^T (T_n(\mu^n)\nabla b^n, \nabla z)\, dt - \int_0^T (b^n \omega^n, z)\, dt \\
&+ \int_0^T \left(\frac{T_k(\mu^n)\,|Dv^n|^2}{1+n^{-1}|Dv^n|^2}, z\right) dt \\
&\to \int_0^T (bv, \nabla z)\, dt - \int_0^T (\mu \nabla b, \nabla z)\, dt - \int_0^T (b\omega, z)\, dt \\
&+ \int_0^T \left(T_k(\mu)\,|Dv|^2, z\right) dt
\end{aligned}$$

for all $z \in L^\infty(0,T,W^{1,q}(\Omega))$, where $q \in (80, \infty]$. This means that

$$\partial_t b^n \rightharpoonup \partial_t b \text{ weakly in } L^1\left(0,T,W^{-1,q}(\Omega)\right) \text{ for all } q \in [1, 80/79).$$

Using Lemma 5.2.3 and (5.65), (5.64), (5.174), (5.175) we deduce that

$$\frac{1}{\sqrt{T_n(\mu^n)}\,(b^n)^{\frac{1+\lambda}{2}}} \to \frac{1}{\sqrt{\mu} b^{\frac{1+\lambda}{2}}} \quad \text{strongly in } L^p(\Omega^T) \text{ for all } 1 \leqslant p < \infty \text{ and for all } 1 \leqslant \lambda < \infty. \tag{5.196}$$

Combining (5.196), (5.186) implies

$$\frac{\sqrt{T^n(\mu^n)}}{(b^n)^{\frac{1+\lambda}{2}}}\nabla b^n \rightharpoonup \frac{\sqrt{\mu}}{(b)^{\frac{1+\lambda}{2}}}\nabla b \qquad \text{weakly in } L^q(\Omega^T) \text{ for all } q \in \left[1, \frac{80}{79}\right).$$

Next, using (5.143) we improve convergence result to

$$\frac{\sqrt{T^n(\mu^n)}}{(b^n)^{\frac{1+\lambda}{2}}}\nabla b^n \rightharpoonup \frac{\sqrt{\mu}}{(b)^{\frac{1+\lambda}{2}}}\nabla b \qquad \text{weakly in } L^2(\Omega^T).$$

Employing the same reasoning one can show that

$$\frac{T^n(\mu^n)}{(b^n)^{\frac{1}{2}}}\nabla b^n \rightharpoonup \frac{\mu}{b^{\frac{1}{2}}}\nabla b \qquad \text{weakly in } L^q(\Omega^T) \text{ for all } q \in \left[1, \frac{80}{79}\right).$$

Finally, thanks to (5.151), (5.175) and Lemma 5.2.3 we get

$$\sqrt{b^n} \to \sqrt{b} \text{ strongly in } L^q\left(0, T, L^q(\Omega)\right) \text{ for all } q \in [1, 10/3).$$

Thus, based on Lemma 5.2.7, we see, that there exists a subsequence (which we do not relabel) such that

$$\sqrt{b^n(t)} \to \sqrt{b(t)} \text{ strongly in } L^q(\Omega) \text{ for all } q \in [1, 10/3) \text{ and almost all times } t \in (0, T).$$

This gives us

$$\left(\sqrt{b^n(t)}, z\right) \to \left(\sqrt{b(t)}, z\right) \text{ for all } \varphi \in \mathcal{D}(\Omega) \text{ for almost all times } t \in (0, T).$$

The obtained convergence results are sufficient to pass to the limit with $n \to \infty$ in equations (5.66)-(5.68) and the inequalities (5.71), (5.143), (5.130), (5.159) to get Theorem 5.2.8.

5.2.8. Proof of Theorem 5.1.1

For an arbitrary k, we denote by (v^k, ω^k, b^k) a solution to the problem (5.45)-(5.47), whose existence was established in Theorem 5.2.8.

k-independent estimates

First, from (5.45) tested with v^k and (5.49), we get

$$\sup_{t\in(0,T)} \|v^k(t)\|_2^2 + \int_{\Omega^T} \left(1 + T_k(\mu^k)\right) |D(v^k)|^2 dxdt \leqslant C. \tag{5.197}$$

Thus, using the Korn inequality and interpolation inequality $\|f\|_{\frac{10}{3}}^{\frac{10}{3}} \leqslant \|f\|_2^{\frac{4}{3}} \|f\|_{1,2}^2$, we get

$$\int_0^T \left\|v^k\right\|_{1,2}^2 + \left\|v^k\right\|_{\frac{10}{3}}^{\frac{10}{3}} dt \leqslant C. \tag{5.198}$$

Moreover, from (5.49) we have

$$\int_0^T \left\|T_k(\mu^k)\right\|_{\frac{8-5\lambda}{3}}^{\frac{8-5\lambda}{3}} dt \leqslant C(\lambda^{-1}). \tag{5.199}$$

Using (5.199), (5.197) and the Hölder inequality, we obtain

$$\begin{aligned} \left\|T_k(\mu^k)Dv^k\right\|_{\Omega^T,\frac{16-5\lambda}{11}} &\leqslant \left\|\sqrt{T_k(\mu^k)}Dv^k\right\|_{\Omega^T,2} \left\|\sqrt{T_k(\mu^k)}\right\|_{\Omega^T,2\frac{16-5\lambda}{6+5\lambda}} \\ &\leqslant \left\|\sqrt{T_k(\mu^k)}Dv^k\right\|_{\Omega^T,2} \left\|T_k(\mu^k)\right\|_{\Omega^T,\frac{16-5\lambda}{6+5\lambda}}^2 \leqslant C(\lambda^{-1}), \end{aligned} \tag{5.200}$$

due to $\frac{16-5\lambda}{6+5\lambda} \in (1, 8/3)$. Notice that from (5.49) and (5.43), it follows that

$$\int_{\Omega^T} |b^k|^{\frac{8-5\lambda}{3}} \leqslant C(\lambda^{-1}). \tag{5.201}$$

Using (5.49), (5.201), (5.43) and the Hölder inequality yields

$$\begin{aligned} \|\nabla b\|_{2-\lambda} &\leqslant \left\|\sqrt{\frac{\mu^k}{(b^k)^{1+\lambda}}}\nabla b^k\right\|_2 \left\|\sqrt{\frac{(b^k)^{1+\lambda}}{\mu^k}}\right\|_{\frac{2(2-\lambda)}{\lambda}} \\ &\leqslant C\left\|\sqrt{\frac{\mu^k}{(b^k)^{1+\lambda}}}\nabla b^k\right\|_2 \|b^k\|_{2-\lambda}^{\frac{\lambda}{2}} \leqslant C(\lambda^{-1}). \end{aligned} \tag{5.202}$$

Notice that for any $\lambda \in (0,1)$ we can find $\lambda_1, \lambda_2 \in (0,1)$ such that $\frac{7}{8-\lambda} = \frac{1}{2-\lambda_1} + \frac{3}{8-5\lambda_2}$. Additionally, $\lambda_1, \lambda_2 \to 0^+$ as $\lambda \to 0^+$. Thus, by the Hölder inequality and (5.43), (5.202), (5.201), we get

$$\left\|\mu^k\nabla b^k\right\|_{\Omega^T,\frac{8-\lambda}{7}} \leqslant \left\|\nabla b^k\right\|_{\Omega^T,2-\lambda_1}\left\|\mu^k\right\|_{\Omega^T,\frac{8-5\lambda_2}{3}} \leqslant C\left(\lambda_1^{-1},\lambda_2^{-1}\right) \leqslant C(\lambda^{-1}). \tag{5.203}$$

From (5.49), (5.202), (5.201) and (5.203) we have

$$\begin{aligned}\sup_{t\in(0,T)}\left(\|b^k(t)\|_1+\|\ln b^k(t)\|_1\right)+\int_0^T\left(\left\|\nabla b^k\right\|_{2-\lambda}^{2-\lambda}+\left\|b^k\right\|_{\frac{8-5\lambda}{3}}^{\frac{8-5\lambda}{3}}\right)dt\\ +\int_0^T\left\|\mu^k\nabla b^k\right\|_{\frac{8-\lambda}{7}}^{\frac{8-\lambda}{7}}dt\leqslant C(\lambda^{-1}).\end{aligned} \tag{5.204}$$

Based on equation (5.46) we have

$$\begin{aligned}\left\|\partial_t b^k\right\|_{-1,\frac{8-\lambda}{7}} &= \sup_{\varphi\in W^{1,\left(\frac{8-\lambda}{7}\right)'}(\Omega):\|\varphi\|_{W^{1,\left(\frac{8-\lambda}{7}\right)'}(\Omega)}=1}\left|\left\langle\partial_t b^k,\varphi\right\rangle\right|\\ &\leqslant \sup_{\substack{\varphi\in W^{1,\frac{8-\lambda}{1-\lambda}}(\Omega)\\ \|\varphi\|_{1,\frac{8-\lambda}{1-\lambda}}=1}}\left[\left|\left(b^kv^k,\nabla\varphi\right)\right|+\left|\left(\mu^kb^k,\nabla\varphi\right)\right|+\left|\left(\omega^kb^k,\varphi\right)\right|+\left|\left(T_k(\mu^k)|Dv^k|^2,\varphi\right)\right|\right].\end{aligned}$$

Let us note that $W^{1,\frac{8-\lambda}{1-\lambda}}(\Omega)\subset W^{1,8}(\Omega)\subset C^{0,\frac{5}{8}}(\Omega)$, and thus, by the Hölder and Young inequalities we have

$$\begin{aligned}\left\|\partial_t b^k\right\|_{-1,\frac{8-\lambda}{7}} &\leqslant \sup_{\substack{\varphi\in W^{1,\frac{8-\lambda}{1-\lambda}}(\Omega)\\ \|\varphi\|_{1,\frac{8-\lambda}{1-\lambda}}=1}}\Bigg[\left\|v^k\right\|_{\frac{10}{3}}\left\|b^k\right\|_{\frac{40}{23}}\|\nabla\varphi\|_8+\left\|\mu^k\nabla b^k\right\|_{\frac{8-\lambda}{7}}\|\nabla\varphi\|_{\frac{8-\lambda}{1-\lambda}}\\ &\qquad +\left\|\omega^k\right\|_\infty\left\|b^k\right\|_1\|\varphi\|_\infty+\left\|T_k(\mu^k)|Dv^k|^2\right\|_1\|\varphi\|_\infty\Bigg]\\ &\leqslant C\left(\left\|v^k\right\|_{\frac{10}{3}}^{\frac{10}{3}}+\left\|b^k\right\|_{\frac{40}{23}}^{\frac{40}{23}}+\left\|\mu^k\nabla b^k\right\|_{\frac{8-\lambda}{7}}^{\frac{8-\lambda}{7}}+\left\|b^k\right\|_1+\left\|T_k(\mu^k)|Dv^k|^2\right\|_1+1\right).\end{aligned}$$

Finally, by (5.204), (5.201), (5.198), (5.197) we get

$$\int_0^T\left\|\partial_t b^k\right\|_{W^{-1,\frac{8-\lambda}{7}}(\Omega)}dt\leqslant C(\lambda^{-1}). \tag{5.205}$$

Next, notice, that (5.49) and (5.43) imply

$$\sup_{t\in(0,T)}\left\|\omega^k(t)\right\|_\infty+\int_{\Omega^T}b^k|\nabla\omega^k|^2dxdt\leqslant C. \tag{5.206}$$

By the Hölder inequality and (5.206), (5.201) we obtain

$$\begin{aligned}\left\|b^k\nabla\omega^k\right\|_{\frac{16-5\lambda}{11}} &\leqslant \left\|\sqrt{b^k}\nabla\omega^k\right\|_2 \left\|\sqrt{b^k}\right\|_{2\frac{16-5\lambda}{6+5\lambda}} \\ &\leqslant \left\|\sqrt{b^k}\nabla\omega^k\right\|_2 \left\|b^k\right\|^2_{\frac{16-5\lambda}{6+5\lambda}} \leqslant C(\lambda^{-1}).\end{aligned} \tag{5.207}$$

Using (5.204), (5.43) and (5.207) we get

$$\left\|\nabla(b^k\omega^k)\right\|_{\frac{16-5\lambda}{11}} \leqslant C(\lambda^{-1}) \tag{5.208}$$

and

$$\left\|\nabla\left(\frac{b^k\omega^k}{b^k+1}\right)\right\|_{\frac{16-5\lambda}{11}} \leqslant C(\lambda^{-1}). \tag{5.209}$$

By (5.208), (5.201), (5.43) we can write

$$\int_0^T \left\|b^k\omega^k\right\|^{\frac{16-5\lambda}{11}}_{W^{1,\frac{16-5\lambda}{11}}(\Omega)} dt \leqslant C(\lambda^{-1}). \tag{5.210}$$

Using the equation (5.47) and inequalities (5.43), (5.198), (5.207), we deduce (in a similar way as in (5.205)) the following:

$$\int_0^T \left\|\partial_t\omega^k\right\|^{\frac{16-5\lambda}{11}}_{W^{-1,\frac{16-5\lambda}{11}}(\Omega)} dt \leqslant C(\lambda^{-1}). \tag{5.211}$$

Reconstruction of the pressure

We will show that there exists pressure $p_k \in L^2(0,T,L^2(\Omega))$ such that

$$\begin{aligned}\langle v^k_{,t}, w\rangle - \left(G_k\left(\left|v^k\right|^2\right) v^k\otimes v^k, \nabla w\right) + \left(T_k\left(\mu^k\right) D(v^k), D(w)\right) &= \left(p^k, \operatorname{div} w\right) \\ \forall w \in W^{1,2}(\Omega) \text{ and almost all } t\in(0,T).&\end{aligned} \tag{5.212}$$

Combining Lemma 5.2.1 with (5.38) and (5.20), (5.22) gives

$$p_1^k = \mathcal{L}(T_k(\mu^k)Dv^k) \in L^2(\Omega) \qquad \text{for almost all } t\in(0,T), \tag{5.213}$$

$$p_2^k = \mathcal{L}(-G_k(|v^k|^2)v^k\otimes v^k) \in L^2(\Omega) \qquad \text{for almost all } t\in(0,T), \tag{5.214}$$

which are uniquely defined for a fixed k. Additionally, using the estimates (5.198), (5.200), we have

$$\left\|p_1^k\right\|_{\frac{16-5\lambda}{11}} \leqslant C \left\|T_k(\mu^k) D v^k\right\|_{\frac{16-5\lambda}{11}} \qquad \text{for almost all } t \in (0,T), \tag{5.215}$$

$$\left\|p_2^k\right\|_{\frac{5}{3}} \leqslant C \left\|v^k\right\|_{\frac{10}{3}}^2 \qquad \text{for almost all } t \in (0,T), \tag{5.216}$$

Moreover, the following equalities hold:

$$\left(p_1^k, \Delta\phi\right) = \left(T_k(\mu^k) D(v^k), \nabla(\nabla\phi)\right) \qquad \text{for all } \phi \in W^{2,2}(\Omega), \tag{5.217}$$

$$\left(p_2^k, \Delta\phi\right) = -\left(G_k\left(\left|v^k\right|^2\right) v^k \otimes v^k, \nabla^2\phi\right) \qquad \text{for all } \phi \in W^{2,2}(\Omega), \tag{5.218}$$

$$\int_\Omega p_1^k dx = \int_\Omega p_2^k dx = 0. \tag{5.219}$$

Let $w \in W^{1,2}(\Omega)$. It can be decomposed (by the Helmholtz decomposition) in the following way:

$$w = \nabla\varphi + \nabla \times A,$$

where $\varphi, A \in W^{2,2}(\Omega)$. Since $\operatorname{div}(\nabla \times A) = 0$, from (5.45) it follows

$$\begin{aligned} \langle v_{,t}^k, \nabla \times A\rangle - \left(G_k\left(\left|v^k\right|^2\right) v^k \otimes v^k, \nabla(\nabla \times A)\right) + \left(T_k\left(\mu^k\right) D(v^k), D(\nabla \times A)\right) \\ = 0. \end{aligned} \tag{5.220}$$

We also see that due to $\operatorname{div} v^k = 0$, we have

$$\langle v_{,t}^k, \nabla\varphi\rangle = 0. \tag{5.221}$$

Since $\operatorname{div}(\nabla \times A) = 0$, we can write

$$\left(p_1^k, \operatorname{div}(\nabla \times A)\right) = 0, \qquad \left(p_2^k, \operatorname{div}(\nabla \times A)\right) = 0. \tag{5.222}$$

Thus, summing (5.222), (5.221), (5.220), (5.218), (5.217), and using the fact that

$$\left(T_k(\mu^k) D(v^k), \nabla(\nabla\phi)\right) = \left(T_k(\mu^k) D(v^k), D(\nabla\phi)\right),$$

we get

$$\langle v^k_{,t}, \nabla\phi + \nabla\times A\rangle - \left(G_k\left(\left|v^k\right|^2\right)v^k\otimes v^k, \nabla\left(\nabla\phi+\nabla\times A\right)\right)$$
$$+\left(T_k\left(\mu^k\right)D(v^k), D\left(\nabla\phi+\nabla\times A\right)\right)\right) = \left(p^k, \operatorname{div}\left(\nabla\phi+\nabla\times A\right)\right),$$

where $p^k = p_1^k + p_2^k$. The obtained equality is exactly (5.212).

Next, using (5.215), (5.216) and estimates (5.198), (5.200), we deduce

$$\int_0^T \left(\|p_1^k\|_{\frac{16-\lambda}{11}}^{\frac{16-5\lambda}{11}} + \|p_2^k\|_{\frac{5}{3}}^{\frac{5}{3}}\right) dt \leqslant C(\lambda^{-1}). \tag{5.223}$$

Now, based on the equation (5.212) for $\lambda \in (0,1)$ and proceeding as in (5.205) we have

$$\left\|\partial v_t^k\right\|_{W^{-1,\frac{16-5\lambda}{11}}(\Omega)} \leqslant C\left(\left\|v^k\right\|_{\frac{10}{3}}^2 + \left\|T_k(\mu^k)Dv^k\right\|_{\frac{16-5\lambda}{11}} + \|p_1\|_{\frac{16-5\lambda}{11}} + \|p_2\|_{\frac{5}{3}}\right). \tag{5.224}$$

Consequently by estimates (5.197), (5.198), (5.223) we have

$$\int_0^T \left\|\partial_t v^k\right\|_{W^{-1,\frac{16-5\lambda}{11}}(\Omega)}^{\frac{16-5\lambda}{11}} dt \leqslant C(\lambda^{-1}). \tag{5.225}$$

Taking the limit $k \to \infty$

By (5.197), (5.198), (5.225), (5.204), (5.205), (5.206), (5.211), (5.201), (5.223), we deduce the existence of a subsequence (which we do not relabel) such that

$$v^k \rightharpoonup^* v \quad \text{weakly* in } L^\infty\left(0,T,L^2_{\operatorname{div}}(\Omega)\right) \cap L^2\left(0,T,W^{1,2}_{\operatorname{div}}(\Omega)\right), \tag{5.226}$$

$$v^k \rightharpoonup v \quad \text{weakly in } L^{\frac{10}{3}}\left(0,T,L^{\frac{10}{3}}(\Omega)\right), \tag{5.227}$$

$$\partial_t v^k \rightharpoonup \partial_t v \quad \text{weakly in } L^q\left(0,T,W^{-1,q}(\Omega)\right) \text{ for all } q \in \left[1,\frac{16}{11}\right), \tag{5.228}$$

$$b^k \rightharpoonup^* b \quad \text{weakly* in } L^q\left(0,T,W^{1,q}(\Omega)\right) \cap L^\infty\left(0,T,L^1(\Omega)\right) \text{ for all } q \in [1,2), \tag{5.229}$$

$$\partial_t b^k \rightharpoonup \partial_t b \quad \text{weakly in } \mathcal{M}\left(0,T,W^{-1,q}(\Omega)\right) \text{ for all } q \in [1,8/7), \tag{5.230}$$

$$\omega^k \rightharpoonup^* \omega \quad \text{weakly* in } L^\infty\left(0,T,L^\infty(\Omega)\right), \tag{5.231}$$

$$\partial_t \omega^k \rightharpoonup \partial_t \omega \quad \text{weakly in } L^q\left(0,T,W^{-1,q}(\Omega)\right) \text{ for all } q \in [1,16/11), \tag{5.232}$$

$$b^k \rightharpoonup b \quad \text{weakly in } L^q\left(0,T,L^q(\Omega)\right) \text{ for all } q \in [1,8/3), \tag{5.233}$$

$$p_1^k \rightharpoonup p_1 \quad \text{weakly in } L^q\left(0,T,L^q(\Omega)\right) \text{ for all } q \in [1,16/11), \tag{5.234}$$

$$p_2^k \rightharpoonup p_2 \quad \text{weakly in } L^{\frac{5}{3}}\left(0,T,L^{\frac{5}{3}}(\Omega)\right). \tag{5.235}$$

From the Aubin-Lions lemma, we conclude that for $\alpha \in (0,1)$

$$v^n \to v \qquad \text{strongly in } L^2\left(0,T,W^{\alpha,2}(\Omega) \cap L^2_{\text{div}}(\Omega)\right), \tag{5.236}$$

$$b^n \to b \qquad \text{strongly in } L^2\left(0,T,W^{\alpha,2}(\Omega)\right). \tag{5.237}$$

We can extract subsequences that converge almost everywhere

$$v^k \to v \qquad \text{almost everywhere in } \Omega^T, \tag{5.238}$$

$$b^k \to b \qquad \text{almost everywhere in } \Omega^T. \tag{5.239}$$

Thus, based on inequalities (5.204), (5.198) and the Vitali Lemma 5.2.3, we have

$$v^k \to v \qquad \text{strongly in } L^q\left(0,T,L^q(\Omega)\right) \text{ for all } q \in [1,10/3), \tag{5.240}$$

$$b^k \to b \qquad \text{strongly in } L^q\left(0,T,L^q(\Omega)\right) \text{ for all } q \in [1,8/3). \tag{5.241}$$

Using (5.210), (5.209), (5.241), (5.231), Lemma 5.2.4 and the uniqueness of the weak limit, we get

$$b^k\omega^k \rightharpoonup b\omega \qquad \text{weakly in } L^q\left(0,T,W^{1,q}(\Omega)\right) \text{ for all } q \in [1,16/11), \tag{5.242}$$

$$\frac{b^k\omega^k}{1+b^k} \rightharpoonup \frac{b\omega}{1+b} \qquad \text{weakly in } L^q\left(0,T,W^{1,q}(\Omega)\right) \text{ for all } q \in [1,16/11). \tag{5.243}$$

Our goal is to strengthen the convergence result for ω^k. To achieve this, we employ the Div-Curl lemma (see Lemma 5.2.2). Let us define two 4-vectors

$$a^k := \left(\omega^k, \omega^k v^k - \mu^k \nabla \omega^k\right), \qquad c^k := \left(b^k\left(1+b^k\right)^{-1}\omega^k, 0, 0, 0\right).$$

Using (5.43), (5.198) and (5.207) yields

$$\left\|a^k\right\|_{L^{\frac{16-5\lambda}{11}}(\Omega^T)} + \left\|c^k\right\|_{L^\infty(\Omega^T)} \leqslant C(\lambda^{-1}).$$

From the equation (5.47), the maximum principle (5.43) and (5.209), we have

$$\left\|\operatorname{div}_{t,x} a^k\right\|_{L^\infty(\Omega^T)} = \left\|\partial_t \omega^k + \operatorname{div}\left(\omega^k v^k\right) - \operatorname{div}\left(\mu^k \nabla \omega^k\right)\right\|_{L^\infty(\Omega^T)} = \kappa_2 \left\|\left(\omega^k\right)^2\right\|_{L^\infty(\Omega^T)} \leqslant C$$

and

$$\left\| \nabla_{t,x} c^k - \left(\nabla_{t,x} c^k\right)^T \right\|_{L^1(\Omega^T)} \leqslant C \left\| \nabla \left(\frac{b^k \omega^k}{1 + b^k} \right) \right\|_{L^1(\Omega^T)} \leqslant C.$$

Using (5.231), (5.240), (5.207), (5.43) in the case of the convergence of a^k and (5.243), (5.43) combined with the uniqueness of the weak limit in the case of the convergence of c^k, we get

$$\begin{aligned} a^k &\rightharpoonup a = \left(\omega, \omega v - \overline{\mu \nabla \omega}\right) && \text{weakly in } L^q(\Omega^T) \text{ for all } q \in [1, 16/11), \\ c^k &\rightharpoonup^* c = \left(b\,(1+b)^{-1}\,\omega, 0, 0, 0\right) && \text{weakly* in } L^\infty(\Omega^T)\ . \end{aligned}$$

Thus, using the Div-Curl Lemma 5.2.2 implies

$$\frac{b^k |\omega^k|^2}{1 + b^k} \rightharpoonup \frac{b|\omega|^2}{1+b} \text{ in the sense of distributions.} \tag{5.244}$$

However, we see that the sequence $\frac{b^k |\omega^k|^2}{1+b^k}$ is bounded in $L^\infty(\Omega^T)$, so a weak sequence can be extracted. Using the uniqueness of the weak limit we get

$$\frac{b^k |\omega^k|^2}{1 + b^k} \rightharpoonup^* \frac{b|\omega|^2}{1+b} \text{ weakly* in } L^\infty(\Omega^T). \tag{5.245}$$

By (5.245) and (5.241) we deduce that

$$\begin{aligned} \int_{\Omega^T} \left(b^k \omega^k\right)^2 dx &= \int_{\Omega^T} b^k \left(b^k + 1\right) \frac{b^k |\omega^k|^2}{1+b^k} dx \\ &\qquad \to \int_{\Omega^T} b\,(b+1)\,\frac{b|\omega|^2}{1+b} dx = \int_{\Omega^T} (b\omega)^2\, dx. \end{aligned} \tag{5.246}$$

From (5.241) and (5.231) it follows

$$b^k \omega^k \rightharpoonup b\omega \text{ weakly in } L^2(\Omega^T). \tag{5.247}$$

And using (5.246) and (5.247) we get

$$b^k \omega^k \to b\omega \text{ strongly in } L^2(\Omega^T). \tag{5.248}$$

Consequently, for a subsequence we have

$$b^k\omega^k \to b\omega \text{ almost everywhere in } \Omega^T. \tag{5.249}$$

By the Vitali Lemma 5.2.3, (5.249), (5.239) and (5.43) we get

$$\omega^k = \frac{b^k\omega^k}{b^k} \to \frac{b\omega}{b} = \omega \text{ strongly in } L^q(\Omega^T) \text{ for all } q \in [1,\infty). \tag{5.250}$$

Having the above convergence and (5.43), it is easy to see that

$$\frac{1}{\omega^k} \to \frac{1}{\omega} \text{ strongly in } L^q(\Omega^T) \text{ for all } q \in [1,\infty). \tag{5.251}$$

Using (5.251) and (5.241), we conclude that

$$\mu^k \to \mu = \frac{b}{\omega} \text{ strongly in } L^q(\Omega^T) \text{ for all } q \in [1,8/3). \tag{5.252}$$

Also, there exists a subsequence (which we do not relabel) such that

$$\mu^k \to \mu \text{ almost everywhere in } \Omega^T. \tag{5.253}$$

From (5.252) combined with (5.229) we deduce

$$\mu^k\nabla b^k \rightharpoonup \mu\nabla b \text{ weakly in } L^q(\Omega^T) \text{ for all } q \in [1,8/7). \tag{5.254}$$

Thanks to (5.197), we can deduce that $\sqrt{T_k(\mu^k)}D(v^k) \rightharpoonup \overline{\sqrt{\mu}D(v)}$ in $L^2(\Omega^T)$, and thus, by (5.226), (5.252) and the uniqueness of the weak limit we have

$$\sqrt{T_k(\mu^k)}D(v^k) \rightharpoonup \sqrt{\mu}D(v) \qquad \text{weakly in } L^2(\Omega^T). \tag{5.255}$$

Again, using (5.253), (5.204), (5.43) and Lemma 5.2.3 we conclude that

$$\sqrt{T_k\left(\mu^k\right)} \to \sqrt{\mu} \text{ strongly in } L^q(\Omega^T) \text{ for all } q \in [1,16/3). \tag{5.256}$$

Now, from (5.256), (5.255) and the weak-strong convergence Lemma 5.2.4 we deduce

$$T_k\left(\mu^k\right) Dv^k \rightharpoonup \mu Dv \text{ weakly in } L^q(\Omega^T) \text{ for all } q \in [1, 16/11). \tag{5.257}$$

Using (5.251), (5.242) and (5.229), we get

$$\mu^k \nabla \omega^k = \frac{\nabla\left(b^k \omega^k\right)}{\omega^k} - \nabla b^k \rightharpoonup \frac{\nabla\left(b\omega\right)}{\omega} - \nabla b \text{ weakly in } L^q(\Omega^T) \text{ for all } q \in [1, 16/11).$$

The convergence results obtained above are sufficient to pass to the limit in (5.45)-(5.47) to obtain (5.15), (5.17), (5.19). Now, we will concentrate on obtaining (5.16). Let us denote by $E^k := |v^k|^2/2 + b^k$. Let us set $w = v^k z$, $z \in W^{1,\infty}(\Omega)$ in (5.212) and sum it with (5.46) to get

$$\begin{aligned} \left\langle E^k_t, z\right\rangle - \left(\left(E^k + p^k\right) v^k, \nabla z\right) + \left(\mu^k \nabla b^k, \nabla z\right) + \left(T_K\left(\mu^k\right) D\left(v^k\right) v^k, \nabla z\right) \\ = \left(-b^k\omega^k, z\right) + \frac{1}{2}\left(\left(2G_k\left(|v^k|^2\right)|v^k|^2 - |v^k|^2 - \Gamma_k\left(|v^k|^2\right)\right) v^k, \nabla z\right). \end{aligned} \tag{5.258}$$

First, let us observe that by (5.198), (5.22), (5.23), the sequence $\left(2G_k\left(|v^k|^2\right)|v^k|^2 - |v^k|^2 - \Gamma_k\left(|v^k|^2\right)\right) v^k$ is bounded in $L^{\frac{10}{9}}(\Omega^T)$, and thus there exists a weakly convergent subsequence (which we do not relabel):

$$\left(2G_k\left(|v^k|^2\right)|v^k|^2 - |v^k|^2 - \Gamma_k\left(|v^k|^2\right)\right) v^k \rightharpoonup \overline{0} \text{ weakly in } L^{\frac{10}{9}}(\Omega^T)\,. \tag{5.259}$$

Using (5.238), (5.22), (5.23), we obtain

$$\left(2G_k\left(|v^k|^2\right)|v^k|^2 - |v^k|^2 - \Gamma_k\left(|v^k|^2\right)\right) v^k \to 0 \text{ almost everywhere in } \Omega^T\,. \tag{5.260}$$

Thus, by (5.260), (5.259) and the Egorov theorem we conclude that

$$\left(2G_k\left(|v^k|^2\right)|v^k|^2 - |v^k|^2 - \Gamma_k\left(|v^k|^2\right)\right) v^k \rightharpoonup 0 \text{ weakly in } L^{\frac{10}{9}}(\Omega^T)\,. \tag{5.261}$$

From (5.238) and (5.239) we have

$$E^k \to E \quad \text{almost everywhere in } \Omega^T. \tag{5.262}$$

Next, (5.198), (5.204) and (5.262) combined with the Egorov theorem yield

$$E^k \rightharpoonup \overline{E} = E \qquad \text{weakly in } L^{\frac{5}{3}}(\Omega^T). \tag{5.263}$$

Now, we see that due to (5.198) and (5.263), $v^k E^k$ is bounded in $L^{\frac{10}{9}}(\Omega^T)$, and thus has a weakly convergent subsequence. Thus, the Egorov lemma, (5.262) and (5.238) imply that

$$v^k E^k \rightharpoonup vE \qquad \text{weakly in } L^{\frac{10}{9}}(\Omega^T)\ . \tag{5.264}$$

Finally, using (5.257), (5.240), (5.234), (5.235) and the weak-strong convergence Lemma 5.2.4, we get

$$T_k(\mu^k)D(v^k)v^k \rightharpoonup \mu D(v)v \qquad \text{weakly in } L^q(\Omega^T) \text{ for all } q \in \left[1, \frac{80}{79}\right), \tag{5.265}$$

$$p_1^k v^k \rightharpoonup p_1 v \qquad \text{weakly in } L^q(\Omega^T) \text{ for all } q \in \left[1, \frac{80}{79}\right), \tag{5.266}$$

$$p_2^k v^k \rightharpoonup p_2 v \qquad \text{weakly in } L^q(\Omega^T) \text{ for all } q \in \left[1, \frac{10}{9}\right). \tag{5.267}$$

From the equation (5.258) and (5.261), (5.264)-(5.267), (5.254), (5.248), recalling that the weakly convergent sequence is bounded, we deduce that

$$\int_0^T \left\|\partial_t E^k\right\|^q_{W^{-1,q}(\Omega)} dt \leqslant C \text{ for all } q \in [1, 80/79).$$

Thus, one can pass to the limit in (5.258) to get (5.16).

Attainment of initial data

In this part, we focus on obtaining initial conditions in a similar fashion as presented in [7]. We start with v. Let us test equation (5.45) with $\varphi \in D(\Omega)$ such that $\operatorname{div} \varphi = 0$ and integrate from 0 to t

$$\left(v^k(t), \varphi\right) - (v_0, \varphi) - \int_0^t \left(v^k \otimes v^k, D\varphi\right) dt + \int_0^t \left(T_k(\mu^k) D v^k, D\varphi\right) dx = 0. \tag{5.268}$$

Using (5.240) and Lemma 5.2.7 we obtain

$$v^k(t) \to v(t) \text{ in } L^2(\Omega) \text{ for almost all } t \in (0,T). \tag{5.269}$$

By (5.257), (5.240) and (5.269), we can pass to the limit in (5.268)

$$(v(t),\varphi) - (v_0,\varphi) - \int_0^t (v\otimes v, D\varphi)\,dt + \int_0^t (\mu Dv, D\varphi)\,dt = 0 \text{ for almost all } t\in(0,T).$$

From this we deduce

$$\lim_{t\to 0^+} (v(t),\varphi) = (v_0,\varphi)\,. \tag{5.270}$$

The equality also holds for $\varphi \in L^2_{\mathrm{div}}(\Omega)$. Indeed, let $\{\varphi_j\}$ be a sequence of smooth functions such that $\varphi_j \to \varphi$ in $L^2(\Omega)$. First, let us observe that by (5.226) we have

$$\lim_{j\to\infty}\lim_{t\to 0^+} |\,(v(t),\varphi_j-\varphi)\,| \leqslant \sup_{t\in(0,T)} \|v(t)\|_2 \lim_{j\to\infty} \|\varphi_j-\varphi\|_2 = 0. \tag{5.271}$$

Now, using (5.270) we have

$$\lim_{j\to\infty}\lim_{t\to 0^+}(v(t),\varphi_j) = \lim_{j\to\infty}\lim_{t\to 0^+}(v(t),\varphi) + \lim_{j\to\infty}\lim_{t\to 0^+}(v(t),\varphi_j-\varphi) = (v_0,\varphi)\,.$$

From this and (5.271) we deduce that (5.270) also holds for φ in $L^2_{\mathrm{div}}(\Omega)$.
Now, testing equation (5.45) with v^k and integrating from 0 to t, we get

$$\|v^k(t)\|_2^2 + 2\int_0^t \left(T_k(\mu^k)Dv^k, Dv^k\right) dt = \|v_0\|_2^2.$$

Next, omitting the second term of the left-hand side and passing to the limit with $k\to\infty$ with the use of (5.269), we obtain

$$\|v(t)\|_2^2 \leqslant \|v_0\|_2^2 \text{ for a.a. } t\in(0,T). \tag{5.272}$$

Using (5.270) and (5.272) we conclude that

$$\begin{aligned}\lim_{t\to 0^+}\|v(t)-v_0\|_2^2 &= \lim_{t\to 0^+}\left(\|v(t)\|_2^2+\|v_0\|_2^2-2\,(v(t),v_0)\right)\\ &\leqslant \|v_0\|_2^2+\|v_0\|_2^2-2\,(v_0,v_0)=0.\end{aligned} \tag{5.273}$$

Similarly, we can show the attainment of initial data for ω.

Now, we will concentrate on showing the attainment of initial data by b. Before we proceed further, we will establish more convergence results. By (5.253), (5.204), (5.43) and Lemma 5.2.3, we have

$$\sqrt{\mu^k} \to \sqrt{\mu} = \sqrt{\frac{b}{\omega}} \text{ strongly in } L^q(\Omega^T) \text{ for all } q \in [1, 16/3). \tag{5.274}$$

From (5.274) combined with (5.229), we deduce

$$\sqrt{\mu^k}\nabla b^k \rightharpoonup \sqrt{\mu}\nabla b \text{ weakly in } L^q(\Omega^T) \text{ for all } q \in [1, 16/11). \tag{5.275}$$

From (5.229), (5.239), Lemma 5.2.3 it follows that

$$\sqrt{b^k} \to \sqrt{b} \text{ strongly in } L^q\left(0,T,L^q(\Omega)\right) \text{ for all } q \in [1,4), \tag{5.276}$$

By (5.276) and Lemma 5.2.7 we get

$$\sqrt{b^k(t)} \to \sqrt{b(t)} \text{ in } L^2(\Omega) \text{ for almost all } t \in (0,T). \tag{5.277}$$

Now, using (5.50) for almost all times $t \in (0,T)$ we have

$$\begin{aligned}&\left(\sqrt{b^k(t)},\varphi\right)-\int_0^t\left(\sqrt{b^k}v^k,\nabla\varphi\right)d\tau+\int_0^t\left(\frac{1}{2\sqrt{\omega^k}}\sqrt{\mu^k}\nabla b^k,\nabla\varphi\right)d\tau\\ &\qquad\geqslant \frac{1}{2}\int_0^t\left(\sqrt{b^k}\omega^k,\varphi\right)d\tau+\left(\sqrt{b_0^k},\varphi\right)\qquad \forall\varphi\in D(\Omega),\varphi\geqslant 0.\end{aligned}$$

Using (5.276), (5.275), (5.240), (5.229), (5.252), (5.277) and letting $k \to \infty$, we obtain

$$\begin{aligned}&\left(\sqrt{b(t)},\varphi\right)-\int_0^t\left(\sqrt{b}v,\nabla\varphi\right)d\tau+\int_0^t\left(\frac{\sqrt{\omega}}{2}\sqrt{\mu}\nabla b,\nabla\varphi\right)d\tau\\ &\geqslant \frac{1}{2}\int_0^t\left(\sqrt{b}\omega,\varphi\right)d\tau+\left(\sqrt{b_0},\varphi\right)\qquad \forall\varphi\in D(\Omega),\varphi\geqslant 0 \text{ for almost all } t\in(0,T).\end{aligned}$$

Finally, letting $t \to 0^+$ gives

$$\liminf_{t\to 0^+} \left(\sqrt{b(t)}, \varphi\right) \geqslant \left(\sqrt{b_0}, \varphi\right) \qquad \forall \varphi \in D(\Omega), \varphi \geqslant 0. \tag{5.278}$$

Note that the obtained inequality is also valid for $\varphi \in L^2(\Omega)$, as before in (5.270), due to the density argument. Now, setting $z = I_{\{0\leqslant\tau\leqslant t\}}$ in (5.258) and integrating from 0 to t, we get

$$\int_0^t \langle \partial_t E^k, 1\rangle d\tau = -\int_0^t \left(b^k\omega^k, 1\right) d\tau.$$

Thus,

$$\int_\Omega b^k(x,t)dx + \int_\Omega |v^k(x,t)|^2 dx = -\int_0^t \left(b^k\omega^k, 1\right) d\tau + \int_\Omega b_0^k(x)dx + \int_\Omega |v_0(x)|^2 dx.$$

Using (5.277), (5.269), (5.248) and letting letting $k \to \infty$, we obtain

$$\int_\Omega b(x,t)dx + \int_\Omega |v(x,t)|^2 dx = -\int_0^t (b\omega, 1)\, d\tau + \int_\Omega b_0(x)dx + \int_\Omega |v_0(x)|^2 dx$$

for almost all $t \in (0,T)$. Finally, letting $t \to 0^+$, we get

$$\limsup_{t\to 0^+} \left(\int_\Omega b(x,t)dx + \int_\Omega |v(x,t)|^2 dx\right) = \int_\Omega b_0(x)dx + \int_\Omega |v_0(x)|^2 dx.$$

Thus, employing (5.273), we get

$$\limsup_{t\to 0^+} \int_\Omega b(x,t)dx = \int_\Omega b_0(x)dx. \tag{5.279}$$

Notice that by (5.279) and (5.278) we have

$$\begin{aligned}
\limsup_{t\to 0^+} \|\sqrt{b(t)} - \sqrt{b_0}\|_2^2 &= \limsup_{t\to 0^+} \left(\|b(t)\|_1 + \|b_0\|_1 - 2\left(\sqrt{b(t)}, \sqrt{b_0}\right)\right) \\
&\leqslant \|b_0\|_1 + \|b_0\|_1 + 2\limsup_{t\to 0^+} \left(-\left(\sqrt{b(t)}, \sqrt{b_0}\right)\right) \\
&\leqslant 2\|b_0\|_1 - 2\liminf_{t\to 0^+} \left(\sqrt{b(t)}, \sqrt{b_0}\right) \\
&\leqslant 2\|b_0\|_1 - 2\left(\sqrt{b_0}, \sqrt{b_0}\right) \leqslant 0.
\end{aligned}$$

Now, by (5.229) it is straightforward to show the attainment of initial data

$$\begin{aligned}\lim_{t\to 0^+} \|b(t)-b_0\|_1 &\leqslant \lim_{t\to 0^+} \left\|\sqrt{b(t)}-\sqrt{b_0}\right\|_2 \left\|\sqrt{b(t)}+\sqrt{b_0}\right\|_2 \\ &\leqslant 2 \sup_{\tau\in(0,T)} \|b(\tau)\|_1^{1/2} \lim_{t\to 0^+} \left\|\sqrt{b(t)}-\sqrt{b_0}\right\|_2 \\ &= 0.\end{aligned}$$

This concludes the proof of the theorem.

Summary

In the thesis, the local-in-time existence of regular solutions has been examined. First, the existence was shown in the case of periodic domains and data from H^2. Moreover, the condition that ensures that obtained local solutions are in fact global was formulated. The basic idea behind the condition is to consider functions with small enough oscillations (measured with the L^2 norm of Laplacian). Next, it was shown that the previous assumption on the regularity of initial data can be relaxed - the local-in-time solution exists provided initial data belongs to H^s, where $s > \frac{d}{2}$. The presented approach to the problem of finding possibly the largest space for which the local-in-time existence holds can be extended to the Besov or Tribel-Lizorkin spaces. The applied methodology would be similar. These results could be also used as the starting point for the considerations of the case of a bounded domain. Next, the analysis of more complicated i.e. nonlinear boundary conditions used in engineering practice could be attempted. Another interesting direction of subsequent research would be the consideration of turbulent flow's interaction with deformable structures (FSI).

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Appendix A

Kato-Ponce commutator estimate in $\mathbb{T}^d$

The main aim of this chapter is to adapt the proof of the classical estimate of J^s commutator from $\mathbb{R}^d$ to $\mathbb{T}^d$. We closely follow the proof given in [26] modifying parts which are different due to the choice of the domain and operators' definitions. We will also use results from functional analysis concerning spaces defined on the torus and from pseudo-differential operator theory.

A.1. Definitions and theorems of pseudo-differential operator theory

To prove Lemma 1.3.3 we will utilise some results from pseudo-differential operator theory. In the following definitions, we introduce the needed apparatus.

Definition A.1.1 (see [9]). Let $m : \mathbb{T}^d \times \mathbb{Z}^{dr} \to \mathbb{C}$ be a measurable function usually referred to as a symbol. Then, the periodic multi-linear pseudo-differential operator associated with a symbol m is the multilinear operator defined by

$$T_m(f)(x) = \sum_{\xi \in \mathbb{Z}^{dr}} e^{i2\pi\langle x, \xi_1+\xi_2+\cdots+\xi_r\rangle} m(x,\xi)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\dots\hat{f}_r(\xi_r),$$

where $x \in \mathbb{T}^d$, $\xi = (\xi_1, \xi_2, \dots, \xi_r)$, $f = (f_1, f_2, \dots, f_r) \in \mathcal{D}(\mathbb{T}^d)^r$ and

$$\hat{f}(\xi_i) = \int_{\mathbb{T}^d} e^{-i2\pi\langle x, \xi_i\rangle} f_i(x)dx.$$

Definition A.1.2 (see Definition 3.3.1 in [42])**.** Let $\sigma : \mathbb{Z}^d \to \mathbb{C}$ and $1 \leqslant i, j \leqslant d$. Let $\delta_j \in \mathbb{N}^d$ be defined by

$$(\delta_j)_i = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

We define the forward difference operator Δ_{ξ_j} by

$$\Delta_{\xi_j}\sigma(\xi) := \sigma(\xi + \delta_j) - \sigma(\xi)$$

and for $\alpha \in \mathbb{N}^d$ we define

$$\Delta_\xi^\alpha := \Delta_{\xi_1}^{\alpha_1} \dots \Delta_{\xi_d}^{\alpha_d}.$$

Theorem A.1.1 (see Theorem 3.3 in [9])**.** *Assume that $m : \mathbb{T}^d \times \mathbb{Z}^{dr} \to \mathbb{C}$ is a measurable function that satisfies the discrete symbol inequalities*

$$\sup_{x \in \mathbb{T}^d} \left|\Delta_{\xi_1}^{\alpha_1}\Delta_{\xi_2}^{\alpha_2} \dots \Delta_{\xi_r}^{\alpha_r} m(x, \xi_1, \xi_2, \dots, \xi_r)\right| \leqslant \frac{C_\alpha}{(1 + |\xi_1|^2 + \dots + |\xi_r|^2)^{\frac{|\alpha|}{2}}} \tag{A.1}$$

for all $|\alpha| = |\alpha_1| + |\alpha_2| + \dots + |\alpha_r| \leqslant \left[\frac{3dr}{2}\right] + 1$. Then, the periodic multi-linear pseudo-differential operator T_m (see Definition A.1.1) extends to a bounded operator from $L^{p_1}(\mathbb{T}^d) \times L^{p_2}(\mathbb{T}^d) \times \dots L^{p_r}(\mathbb{T}^d)$ into $L^p(\mathbb{T}^d)$ provided that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \dots \frac{1}{p_r}, \quad 1 < p < \infty, \quad 1 < p_i \leqslant \infty.$$

Lemma A.1.2 (see Corollary 4.5.7 in [42])**.** *Let $0 < \delta \leqslant 1$, $0 \leqslant \rho \leqslant 1$, $m \in \mathbb{R}$. Let $a : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{C}$ satisfy*

$$\left|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)\right| \leqslant \frac{C_{a\alpha\beta m}}{(1 + |\xi|^2)^{\frac{m + \rho|\alpha| - \delta|\beta|}{2}}} \quad \forall x \in \mathbb{T}^d, \xi \in \mathbb{R}^d \tag{A.2}$$

for $|\alpha| \leqslant N_1$ and $|\beta| \leqslant N_2$. Then the restriction $\overline{a} = a|_{\mathbb{T}^d \times \mathbb{Z}^d}$ satisfies the estimate

$$\left|\Delta_\xi^\alpha \partial_x^\beta \overline{a}(x, \xi)\right| \leqslant \frac{C'_{a\alpha\beta m} C_{a\alpha\beta m}}{(1 + |\xi|^2)^{\frac{m + \rho|\alpha| - \delta|\beta|}{2}}} \quad \forall x \in \mathbb{T}^d, \xi \in \mathbb{Z}^d$$

for $|\alpha| \leqslant N_1$ *and* $|\beta| \leqslant N_2$.

In the proof of Lemma 1.3.3 we will need some results from the interpolation theory. First, let us introduce the needed definitions.

Definition A.1.3. We define a complex strip in the following way:

$$S = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < 1\}.$$

Definition A.1.4 (see [21])**.** A continuous function $F : \overline{S} \to \mathbb{C}$, which is analytic in S is said to be of admissible growth if there is $0 \leqslant \alpha < \pi$ such that

$$\sup_{z \in \overline{S}} \frac{\log |F(z)|}{e^{\alpha |\operatorname{Im} z|}} < \infty$$

Definition A.1.5 (see [21])**.** Let (Ω, Σ, μ) be a measure space and let $\mathcal{X}_1, ..., \mathcal{X}_m$ be linear spaces. Let us assume that for every $z \in \overline{S}$ there is a linear operator $T_z : \mathcal{X}_1 \times \ldots \mathcal{X}_m \to \overline{L}^0(\mu)$, where $\overline{L}^0(\mu)$ denotes the space of all equivalence classes of complex-valued measurable functions on Ω with the topology of convergence in measure on μ-finite sets. The family $\{T_z\}_{z \in \overline{S}}$ is said to be analytic if for any $(x_1, ..., x_m) \in \mathcal{X}_1 \times \ldots \mathcal{X}_m$ and for almost every $\omega \in \Omega$ the function

$$\overline{S} \ni z \longmapsto T_z(x_1, ..., x_m)(\omega), \tag{A.3}$$

is analytic in S and continuous in $\overline{S}$. Additionally, if for $j = 0, 1$ the function

$$\mathbb{R} \times \Omega \ni (t, \omega) \longmapsto T_{j+it}(x_1, ..., x_m)(\omega)$$

is $(\mathcal{L} \times \Sigma)$-measurable for every $(x_1, ..., x_m) \in \mathcal{X}_1 \times \ldots \mathcal{X}_m$, and for almost every $\omega \in \Omega$ the function (A.3) is of admissible growth, then the family $\{T_z\}_{z \in \overline{S}}$ is said to be an admissible analytic family.

The theorem we are about to cite is more general than the stated below. The statement has been adapted to better fit the case at hand.

Theorem A.1.3 (see Theorem 4.1 in [21]). *For $1 \leqslant k \leqslant m$, fix $1 < q_0, q_1, q_{0k}, q_{1k} < \infty$ and for $0 < \theta < 1$ define q, q_k by setting*

$$\frac{1}{q_k} = \frac{1-\theta}{q_{0k}} + \frac{\theta}{q_{1k}}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Assume that $\mathcal{X}_k$ is a dense linear subspace of $L^{q_{0k}}(\mathbb{T}^d) \cap L^{q_{1k}}(\mathbb{T}^d)$ and that $\{T_z\}_{z \in \overline{S}}$ is an admissible analytic family of multilinear operators $T_z : \mathcal{X}_1 \times \cdots \times \mathcal{X}_m \to L^{q_0}(\mathbb{T}^d) \cap L^{q_1}(\mathbb{T}^d)$. Suppose that for every $(h_1, ..., h_m) \in \mathcal{X}_1 \times \dots \mathcal{X}_m$, $t \in \mathbb{R}$ and $j = 0, 1$, we have

$$\|T_{j+it}(h_1, ..., h_m)\|_{L^{q_j}(\mathbb{T}^d)} \leqslant K_j(t) \, \|h_1\|_{L^{q_{j1}}(\mathbb{T}^d)} \dots \|h_m\|_{L^{q_{jm}}(\mathbb{T}^d)}, \tag{A.4}$$

where K_j are Lebesgue measurable functions such that $K_j \in L^1(P_j(\theta, \cdot)dt)$ for all $\theta \in (0,1)$, where

$$P_j(x+iy, t) = \frac{e^{-\pi(t-y)} \sin \pi x}{\sin^2 \pi x + (\cos \pi x - (-1)^j e^{-\pi(t-y)})^2}, \qquad x + iy \in \overline{S}.$$

Then for all $(f_1, ..., f_m) \in \mathcal{X}_1 \times \dots \mathcal{X}_m$, $0 < \theta < 1$, and $s \in \mathbb{R}$ we have

$$\|T_{\theta+is}(f_1, ..., f_m)\|_{L^q(\mathbb{T}^d)} \leqslant \left(\frac{q_0}{q_0 - 1}\right)^{1-\theta} \left(\frac{q_1}{q_1-1}\right)^{\theta} K_\theta(s) \prod_{j=1}^{m} \|f_j\|_{L^{q_j}(\mathbb{T}^d)},$$

where

$$\log K_\theta(s) = \int_{\mathbb{R}} P_0(\theta, t) \log K_0(t+s) dt + \int_{\mathbb{R}} P_1(\theta, t) \log K_1(t+s) dt.$$

Remark A.1.4. *For fixed $x \in (0,1)$ and $y \in \mathbb{R}$ there exists constant $C_{x,y} > 0$ such that*

$$|P_j(x+iy, t)| \leqslant C_{x,y} e^{-\pi|t|} \quad \forall t \in \mathbb{R}.$$

A.2. Proof of Lemma 1.3.3

The presented proof follows the original proof in the work of Kato and Ponce [26]. Some of the more calculation-focused lemmas were moved to Section A.3 to provide a clearer argument. Additionally, in the presented proof term $4\pi^2$ will be omitted in the definition of J^s to shorten a bit the obtained formulas.

Proof of Theorem 1.3.3. For smooth functions, any considered infinite series in this proof will be convergent as the following holds

$$\forall \phi \in C^{\infty}(\mathbb{T}^d) \quad \forall m \in \mathbb{N} \quad \exists C_{\phi,m} \text{ such that } \forall \xi \in \mathbb{Z}^d \quad |\hat{\phi}(\xi)| \leqslant \frac{C_{\phi,m}}{(1+|\xi|^2)^{m/2}}. \tag{A.5}$$

Let us start the proof by rewriting the expression under the norm using Definition 1.3.5:

$$\begin{aligned} J^s(fg)(x) - fJ^s(g)(x) = &\sum_{\xi \in \mathbb{Z}^d} e^{i2\pi\langle x,\xi\rangle} \left(1+|\xi|^2\right)^{s/2} \widehat{fg}(\xi) \\ &- f(x) \sum_{\eta \in \mathbb{Z}^d} e^{i2\pi\langle x,\eta\rangle} \left(1+|\eta|^2\right)^{s/2} \hat{g}(\eta). \end{aligned}$$

Now, we use the fact that the Fourier transform of a product is a convolution of transforms

$$\begin{aligned} J^s(fg)(x) - fJ^s(g)(x) = &\sum_{\xi \in \mathbb{Z}^d} e^{i2\pi\langle x,\xi\rangle} \left(1+|\xi|^2\right)^{s/2} \sum_{\eta \in \mathbb{Z}^d} \hat{f}(\eta)\hat{g}(\xi-\eta) \\ &- \sum_{\xi \in \mathbb{Z}^d} e^{i2\pi\langle x,\xi\rangle} \hat{f}(\xi) \sum_{\eta \in \mathbb{Z}^d} e^{i2\pi\langle x,\eta\rangle} \left(1+|\eta|^2\right)^{s/2} \hat{g}(\eta). \end{aligned}$$

We change the variables in the first integral on the right-hand side $\overline{\xi} = \xi - \eta$:

$$\begin{aligned} J^s(fg)(x) - fJ^s(g)(x) = &\sum_{\eta \in \mathbb{Z}^d} \sum_{\overline{\xi} \in \mathbb{Z}^d} e^{i2\pi\langle x,\overline{\xi}+\eta\rangle} \left(1+|\overline{\xi}+\eta|^2\right)^{s/2} \hat{f}(\eta)\hat{g}(\overline{\xi}) \\ &- \sum_{\eta \in \mathbb{Z}^d} \sum_{\xi \in \mathbb{Z}^d} e^{i2\pi\langle x,\xi+\eta\rangle} \left(1+|\eta|^2\right)^{s/2} \hat{f}(\xi)\hat{g}(\eta). \end{aligned}$$

We can rewrite this in the following way

$$\begin{aligned} &J^s(fg)(x) - fJ^s(g)(x) \\ &\quad = \sum_{\eta \in \mathbb{Z}^d} \sum_{\xi \in \mathbb{Z}^d} e^{i2\pi\langle x,\xi+\eta\rangle} \left(\left(1+|\xi+\eta|^2\right)^{s/2} - \left(1+|\eta|^2\right)^{s/2}\right) \hat{f}(\xi)\hat{g}(\eta). \end{aligned} \tag{A.6}$$

Now, we aim to rewrite the obtained expression as a sum of three terms. To do this we introduce the following partition of unity: let $\{\Phi_j\}_{j=1}^3 \subset C^{\infty}(\mathbb{R})$ be such that

$$0 \leqslant \Phi_j \leqslant 1 \quad \text{for } j = 1,2,3,$$

$$\Phi_1 + \Phi_2 + \Phi_3 = 1 \text{ on } [0,\infty),$$

$$\operatorname{supp}\Phi_1 \subset \left(-\frac{1}{9}, \frac{1}{9}\right), \qquad \operatorname{supp}\Phi_2 \subset \left(\frac{1}{10}, 10\right), \qquad \operatorname{supp}\Phi_3 \subset (9, \infty). \tag{A.7}$$

The value $-\frac{1}{9}$ in the definition of the Φ_1 actually can be replaced by any negative value. Now, we can write

$$J^s(fg)(x) - fJ^s(g)(x) = \sum_{j=1}^{3} \sigma_j(D)(f,g)(x), \tag{A.8}$$

where

$$\sigma_j(D)(f,g)(x) = \sum_{\eta\in\mathbb{Z}^d}\sum_{\xi\in\mathbb{Z}^d} e^{i2\pi\langle x,\xi+\eta\rangle}\sigma_j(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)$$

and

$$\sigma_j(\xi,\eta) = \left(\left(1+|\xi+\eta|^2\right)^{s/2} - \left(1+|\eta|^2\right)^{s/2}\right)\Phi_j\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right).$$

Let us note that notation analogous to $\sigma_j(D)(f,g)$ will be used in later parts for different symbols. Now, we aim to provide the estimate for each term $\sigma_j(D)(f,g)$. For the reader's convenience, each estimate will be obtained in a separate subsection.

A.2.1. Step 1: Estimate of $\sigma_1(D)(f,g)$

We start with the transforming function $\sigma_1(\xi,\eta)$ in the following way

$$\begin{aligned}\sigma_1(\xi,\eta) &= \left(1+|\eta|^2\right)^{s/2}\left(\left(\frac{1+|\xi+\eta|^2}{1+|\eta|^2}\right)^{s/2} - 1\right)\Phi_1\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)\\ &= \left(1+|\eta|^2\right)^{s/2}\left(\left[1+(1+|\eta|^2)^{-1}\langle\xi,\xi+2\eta\rangle\right]^{s/2} - 1\right)\Phi_1\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right).\end{aligned}$$

Our goal is to show that σ_1 after some transformations satisfies condition (A.1). However, checking condition (A.1) can be troublesome, and instead, we will verify condition (A.2) and use Lemma A.1.2. It is easier to check first condition (A.2) for (ξ,η) such that $\frac{1+|\xi|^2}{1+|\eta|^2} < \frac{1}{9}$ (compare with (A.7)). Now, we perform the Taylor expansion of the term $[1+(1+|\eta|^2)^{-1}\langle\xi,\xi+2\eta\rangle]^{s/2}$. To do this we recall that $(1+x)^\alpha = \sum_{i=0}^{\infty}\binom{\alpha}{i}x^i$ for $|x|<1$, where $\binom{\alpha}{i} = \prod_{r=1}^{i}\frac{\alpha-r+1}{r}$, $\binom{\alpha}{0} = 1$. Indeed, based on Lemma A.3.1 from Section A.3 and the fact that $\frac{1+|\xi|^2}{1+|\eta|^2} < \frac{1}{9}$ we have that $|(1+|\eta|^2)^{-1}\langle \xi, \xi+2\eta\rangle| < 1$. Thus we may write

$$\begin{aligned}\sigma_1(\xi,\eta) &= \left(1+|\eta|^2\right)^{s/2}\left[\sum_{r=0}^{\infty}\binom{s/2}{r}(1+|\eta|^2)^{-r}\langle \xi,\xi+2\eta\rangle^r - 1\right]\Phi_1\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)\\ &= \sum_{r=1}^{\infty}\binom{s/2}{r}(1+|\eta|^2)^{s/2-r}\langle \xi,\xi+2\eta\rangle^r\Phi_1\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right).\end{aligned}$$

Now, we aim to reformulate the terms under the sum. Let us recall that

$$(J^{s-1}g)\hat{}(\eta) = (1+|\eta|^2)^{\frac{s-1}{2}}\hat{g}(\eta), \qquad (\partial f)\hat{}(\xi) = \xi\hat{f}(\xi). \tag{A.9}$$

Thus, for (ξ,η) such that $\frac{1+|\xi|^2}{1+|\eta|^2} < \frac{1}{9}$ we may write:

$$\begin{aligned}\hat{f}(\xi)\hat{g}(\eta)\sigma_1(\xi,\eta) &= \sum_{r=1}^{\infty}\langle \sigma_{1,r}, (\partial f)\hat{}(\xi)\rangle (J^{s-1}g)\hat{}(\eta)\\ &\equiv \langle \overline{\sigma}_1(\xi,\eta), (\partial f)\hat{}(\xi)\rangle (J^{s-1}g)\hat{}(\eta),\end{aligned} \tag{A.10}$$

where

$$\sigma_{1,r}(\xi,\eta) = \binom{s/2}{r}(1+|\eta|^2)^{-r+1/2}\langle \xi,\xi+2\eta\rangle^{r-1}(\xi+2\eta)\Phi_1\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right). \tag{A.11}$$

For (ξ,η) such that $\frac{1+|\xi|^2}{1+|\eta|^2} \geqslant \frac{1}{9}$ things are simpler:

$$\begin{aligned}\sigma_1(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta) &= 0\cdot\hat{f}(\xi)\hat{g}(\eta) = \langle 0, (\partial f)\hat{}(\xi)\rangle (J^{s-1}g)\hat{}(\eta)\\ &\equiv \langle \overline{\sigma}_1(\xi,\eta), (\partial f)\hat{}(\xi)\rangle (J^{s-1}g)\hat{}(\eta).\end{aligned}$$

With this, we can conclude that

$$\overline{\sigma}_1(\xi,\eta) = \begin{cases}\sum_{r=1}^{\infty}\sigma_{1,r}(\xi,\eta) & \text{for } (\xi,\eta) : \frac{1+|\xi|^2}{1+|\eta|^2} < \frac{1}{9}\\ 0 & \text{otherwise}\end{cases}. \tag{A.12}$$

As mentioned before, we will show that for each r function $\sigma_{1,r}$ fulfils condition (A.2) up to some number of differentiations $k(d) \in \mathbb{N}$. We will analyse $\sigma_{1,r}$ step by step. Let $m \geqslant 0$. Then, based on Lemma A.3.3 from Section A.3 let us observe that for $\alpha_i \in \mathbb{N}$ such that

$\sum_{i=1}^{d} \alpha_i = m$ we have

$$\left| \frac{\partial^m \left[(1+|\eta|^2)^{-r+1/2}\right]}{\partial_{\eta_1}^{\alpha_1} \dots \partial_{\eta_d}^{\alpha_d}} \right| \leqslant C(1+r)^m (1+|\eta|^2)^{-r+1/2-\frac{m}{2}} \qquad \forall \eta \in \mathbb{R}^d.$$

Using the assumption $\frac{1+|\xi|^2}{1+|\eta|^2} < \frac{1}{9}$ we can write

$$\left| \frac{\partial^m \left[(1+|\eta|^2)^{-r+1/2}\right]}{\partial_{\eta_1}^{\alpha_1} \dots \partial_{\eta_d}^{\alpha_d}} \right| \leqslant \frac{C(1+r)^m}{(1+|\xi|^2+|\eta|^2)^{\frac{m}{2}}(1+|\eta|^2)^{r-1/2}} \quad \forall (\xi,\eta) : \frac{1+|\xi|^2}{1+|\eta|^2} < \frac{1}{9}. \tag{A.13}$$

Now let us focus on the term $\langle \xi, \xi+2\eta \rangle^{r-1}(\xi_k + 2\eta_k)$. Let $\alpha_i, \beta_i \in \mathbb{N}$ be such that $\sum_{i=1}^{d}(\alpha_1+\beta_i) = m$. Based on Lemma A.3.4 from Section A.3 we have

$$\begin{aligned} &\left| \frac{\partial^m \left[\langle \xi, \xi+2\eta \rangle^{r-1}(\xi_k+2\eta_k)\right]}{\partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_d}^{\alpha_d} \partial_{\eta_1}^{\beta_1} \dots \partial_{\eta_d}^{\beta_d}} \right| \\ &\qquad \leqslant C\,(1+r)^m \left(\frac{7}{9}\right)^r \frac{(1+|\eta|^2)^{r-\frac{1}{2}}}{(1+|\xi|^2+|\eta|^2)^{\frac{m}{2}}} \quad \forall (\xi,\eta) : \frac{1+|\xi|^2}{1+|\eta|^2} < \frac{1}{9}. \end{aligned} \tag{A.14}$$

Now we will handle the last term in the definition of $\sigma_{1,r}$. From Lemma A.3.5 from Section A.3 we have that

$$\left| \frac{\partial^m \left[\Phi_1\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)\right]}{\partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_d}^{\alpha_d} \partial_{\eta_1}^{\beta_1} \dots \partial_{\eta_d}^{\beta_d}} \right| \leqslant \frac{C}{(1+|\xi|^2+|\eta|^2)^{\frac{m}{2}}} \qquad \forall (\xi,\eta) \in \mathbb{R}^{2d}. \tag{A.15}$$

Thus based on (A.11), (A.12), (A.13), (A.14), (A.15) and Lemma A.3.2 from Section A.3 we can finally write

$$\left| \frac{\partial^m \overline{\sigma}_1(\eta,\xi)}{\partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_d}^{\alpha_d} \partial_{\eta_1}^{\beta_1} \dots \partial_{\eta_d}^{\beta_d}} \right| \leqslant \frac{C(m)}{(1+|\xi|^2+|\eta|^2)^{\frac{m}{2}}} \sum_{r=1}^{\infty} \left| \binom{s/2}{r} \right| (1+r)^m \left(\frac{7}{9}\right)^r.$$

The sum on the right-hand side is finite based on the D'Alembert criterion for series convergence. Indeed we see that

$$\frac{\left|\binom{s/2}{r+1}\right| (2+r)^m \left(\frac{7}{9}\right)^{r+1}}{\left|\binom{s/2}{r}\right| (1+r)^m \left(\frac{7}{9}\right)^r} = \frac{7}{9} \left| \frac{s/2-(r+1)+1}{r+1} \right| \left(\frac{r+2}{r+1}\right)^m \overset{r\to\infty}{\longrightarrow} \frac{7}{9} < 1.$$

Based on Lemma A.1.2 we have

$$\left|\Delta_{\xi_1}^{\alpha_1} \ldots \Delta_{\xi_d}^{\alpha_d} \Delta_{\eta_1}^{\beta_1} \ldots \Delta_{\eta_d}^{\beta_d} \overline{\sigma}_1(\xi,\eta)\right| \leqslant \frac{C(m,s)}{(1+|\xi|^2+|\eta|^2)^{\frac{m}{2}}}.$$

From (A.10), (A.12) and Theorem A.1.1 we have

$$\|\sigma_1(D)(f,g)\|_p = \left\|\overline{\sigma}_1(D)(\partial f, J^{s-1}g)\right\|_p \leqslant C\,\|\partial f\|_{p_1}\left\|J^{s-1}g\right\|_{p_2}. \tag{A.16}$$

A.2.2. Step 2: Estimate of $\sigma_3(D)(f,g)$

Now we will consider the term $\sigma_3(D)(f,g)$. Firstly, we define

$$\sigma_{3,1}(\xi,\eta) = \left(\left(1+|\xi+\eta|^2\right)^{s/2} - 1\right)\Phi_3\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right) \tag{A.17}$$

and

$$\sigma_{3,2}(\xi,\eta) = \left(\left(1+|\eta|^2\right)^{s/2} - 1\right)\Phi_3\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right). \tag{A.18}$$

We clearly see that $\sigma_3 = \sigma_{3,1} - \sigma_{3,2}$. Based on (A.9) it follows

$$\sigma_{3,1}(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta) = \frac{(1+|\xi+\eta|^2)^{s/2}-1}{(1+|\xi|^2)^{s/2}}\Phi_3\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)(J^s f)\hat{}(\xi)\hat{g}(\eta). \tag{A.19}$$

Now we have to show that

$$\sigma_{3,1}^{*}(\xi,\eta) = \left(1+|\xi|^2\right)^{-s/2}\left(\left(1+|\xi+\eta|^2\right)^{s/2} - 1\right)\Phi_3\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right) \tag{A.20}$$

fulfils condition (A.1). As previously, we will show that condition (A.2) holds and deduce (A.1) from Lemma A.1.2. As before we will split our considerations into two cases: for (ξ,η) such that $\frac{1+|\xi|^2}{1+|\eta|^2} > 9$ and the opposite. We start with the prior case. Based on Lemma A.3.3 from Section A.3 for $\alpha_i \in \mathbb{N}$ such that $\sum_{i=1}^d \alpha_i = m$ we can deduce the following

$$\left|\frac{\partial^m\left[(1+|\xi|^2)^{-s/2}\right]}{\partial_{\xi_1}^{\alpha_1}\ldots\partial_{\xi_d}^{\alpha_d}}\right| \leqslant C(s,m)(1+|\xi|^2)^{-s/2}\left(1+|\xi|^2\right)^{-\frac{m}{2}} \qquad \forall \xi \in \mathbb{R}^d.$$

Using the assumption that $\frac{1+|\xi|^2}{1+|\eta|^2} > 9$ we have

$$\left|\frac{\partial^m \left[(1+|\xi|^2)^{-s/2}\right]}{\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_d}^{\alpha_d}}\right| \leqslant C(1+|\xi|^2)^{-s/2}\left(1+|\xi|^2+|\eta|^2\right)^{-\frac{m}{2}} \qquad \forall(\xi,\eta) : \frac{1+|\xi|^2}{1+|\eta|^2} > 9. \tag{A.21}$$

By using Lemma A.3.6 from Section A.3 for $\alpha_i, \beta_i \in \mathbb{N}$ such that $\sum_{i=1}^d(\alpha_i+\beta_i) = m$ we have

$$\left|\frac{\partial^m \left[(1+|\xi+\eta|^2)^{s/2}-1\right]}{\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_d}^{\alpha_d}\partial_{\eta_1}^{\beta_1}\dots\partial_{\eta_d}^{\beta_d}}\right| \leqslant C(1+|\xi|^2+|\eta|^2)^{\frac{s}{2}-\frac{m}{2}} \qquad \forall(\xi,\eta) : \frac{1+|\xi|^2}{1+|\eta|^2} > 9. \tag{A.22}$$

By employing Lemma A.3.5 from Section A.3 we get

$$\left|\frac{\partial^m \left[\Phi_3\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)\right]}{\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_d}^{\alpha_d}\partial_{\eta_1}^{\beta_1}\dots\partial_{\eta_d}^{\beta_d}}\right| \leqslant \frac{C}{(1+|\xi|^2+|\eta|^2)^{\frac{m}{2}}}. \tag{A.23}$$

Collecting (A.20), (A.21), (A.22) and (A.23) and by using Lemma A.3.2 from Section A.3 we get

$$\left|\frac{\partial^m \sigma_{3,1}^*}{\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_d}^{\alpha_d}\partial_{\eta_1}^{\beta_1}\dots\partial_{\eta_d}^{\beta_d}}\right| \leqslant C\left(\frac{1+|\xi|^2+|\eta|^2}{1+|\xi|^2}\right)^{s/2}\left(1+|\eta|^2+|\xi|^2\right)^{-\frac{m}{2}} \ \ \forall(\xi,\eta) : \frac{1+|\xi|^2}{1+|\eta|^2} > 9.$$

We see that thanks to $\frac{1+|\xi|^2}{1+|\eta|^2} > 9$ we have

$$\frac{1+|\xi|^2+|\eta|^2}{1+|\xi|^2} \leqslant 1 + \frac{|\eta|^2}{1+|\xi|^2} \leqslant 1 + \frac{1}{9} = \frac{10}{9}$$

and thus

$$\left|\frac{\partial^m \sigma_{3,1}^*}{\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_d}^{\alpha_d}\partial_{\eta_1}^{\beta_1}\dots\partial_{\eta_d}^{\beta_d}}\right| \leqslant C\left(1+|\eta|^2+|\xi|^2\right)^{-\frac{m}{2}} \qquad \forall(\xi,\eta) : \frac{1+|\xi|^2}{1+|\eta|^2} > 9.$$

The obtained formula is also valid in the case of $\frac{1+|\xi|^2}{1+|\eta|^2} \leqslant 9$ since $\operatorname{supp}\Phi_3 \subset (9,\infty)$. Thus based on Lemma A.1.2 we can deduce

$$\left|\Delta_{\xi_1}^{\alpha_1} \dots \Delta_{\xi_d}^{\alpha_d} \Delta_{\eta_1}^{\beta_1} \dots \Delta_{\eta_d}^{\beta_d} \sigma_{3,1}^*(\xi,\eta)\right| \leqslant \frac{C_\alpha}{(1+|\xi|^2+|\eta|^2)^{\frac{|\alpha|+|\beta|}{2}}}$$

for all $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}^d$. Finally from (A.19) and Theorem A.1.1 we get

$$\|\sigma_{3,1}(D)(f,g)\|_p = \left\|\sigma_{3,1}^*(D)(J^s f, g)\right\|_p \leqslant C\,\|g\|_{p_3}\,\|J^s f\|_{p_4}\,. \tag{A.24}$$

Now let us proceed with $\sigma_{3,2}$. Before we start let us observe that $\sigma_{3,2}(\xi,0) \equiv 0$. Let us define auxiliary smooth function Ψ such that $0 \leqslant \Psi(x) \leqslant 1$, $\Psi(x) = 1$ for $x < 3/4$, $\Psi(x) = 0$ for $x > 9/10$. Then we can rewrite (A.18) in a following form

$$\begin{aligned}&\sigma_{3,2}(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)\\ &= \begin{cases} |\xi|^{-2}\left\langle \xi, (\partial f)\hat{}(\xi)\right\rangle \left\langle \eta, (GJ^{s-1}g)\hat{}(\eta)\right\rangle \Phi_3\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right) & \text{for } (\xi,\eta) : \frac{1+|\xi|^2}{1+|\eta|^2} > 9 \\ \left\langle 0, (\partial f)\hat{}(\xi)\right\rangle \left\langle 0, (GJ^{s-1}g)\hat{}(\eta)\right\rangle & \text{otherwise} \end{cases},\end{aligned} \tag{A.25}$$

where

$$(Gh)\hat{}(\eta) = g(\eta)\hat{h}(\eta), \tag{A.26}$$

$$g(\eta) = \begin{cases} \eta|\eta|^{-2}\frac{\left(1+|\eta|^2\right)^{s/2}-1}{(1+|\eta|^2)^{s/2-1/2}}\Psi\left(\frac{1}{1+|\eta|^2}\right) & \text{for } \eta : \frac{1}{1+|\eta|^2} < \frac{9}{10} \\ 0 & \text{otherwise} \end{cases}. \tag{A.27}$$

The purpose of the term $\Psi\left(\frac{1}{1+|\eta|^2}\right)$ is to cut-off region near $\eta = 0$, without affecting values for $\eta \in \mathbb{Z}^d\backslash\{0\}$. We see that in view of Lemma A.3.5 from Section A.3 we have

$$\left|\frac{\partial^m\left[\Phi_3\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)\right]}{\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_d}^{\alpha_d}\partial_{\eta_1}^{\beta_1}\dots\partial_{\eta_d}^{\beta_d}}\right| \leqslant \frac{C}{(1+|\eta|^2+|\xi|^2)^{\frac{m}{2}}}. \tag{A.28}$$

With the use of that fact that $\frac{1+|\xi|^2}{1+|\eta|^2} > 9$ (which implies that $|\xi| > 2\sqrt{2}$), we also see that

$$\left|\frac{\partial^m\left[|\xi|^{-2}\xi_j\right]}{\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_d}^{\alpha_d}\partial_{\eta_1}^{\beta_1}\dots\partial_{\eta_d}^{\beta_d}}\right| \leqslant C|\xi|^{-1-m} \leqslant \frac{C}{\left(1+|\eta|^2+|\xi|^2\right)^{\frac{m+1}{2}}}. \tag{A.29}$$

We see that

$$\begin{aligned}&\frac{\partial^m\left[|\xi|^{-2}\eta_k\xi_j\Phi_3\right]}{\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_d}^{\alpha_d}\partial_{\eta_1}^{\beta_1}\dots\partial_{\eta_d}^{\beta_d}}\\ &\quad=\eta_k\frac{\partial^m\left[|\xi|^{-2}\xi_j\Phi_3\right]}{\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_d}^{\alpha_d}\partial_{\eta_1}^{\beta_1}\dots\partial_{\eta_d}^{\beta_d}}+\binom{\beta_k}{1}\frac{\partial^{m-1}\left[|\xi|^{-2}\xi_j\Phi_3\right]}{\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_d}^{\alpha_d}\partial_{\eta_1}^{\beta_1}\dots\partial_{\eta_{k-1}}^{\beta_{k-1}}\partial_{\eta_k}^{\beta_k-1}\partial_{\eta_{k+1}}^{\beta_{k+1}}\dots\partial_{\eta_d}^{\beta_d}}.\end{aligned}$$

Thus we see that by (A.28), (A.29) and Lemma A.3.2 from Section A.3 we can calculate

$$\left|\frac{\partial^m\left[|\xi|^{-2}\eta_k\xi_j\Phi_3\right]}{\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_d}^{\alpha_d}\partial_{\eta_1}^{\beta_1}\dots\partial_{\eta_d}^{\beta_d}}\right| \leqslant \frac{C}{\left(1+|\eta|^2+|\xi|^2\right)^{\frac{m}{2}}}.$$

We see that based on the above, (A.25), Lemma A.1.2 and Theorem A.1.1 we may conclude

$$\left\|\sigma_{3,2}(D)(f,g)\right\|_p \leqslant C\left\|\partial f\right\|_{p_1}\left\|GJ^{s-1}g\right\|_{p_2}. \tag{A.30}$$

Now we need to derive the estimate for $\left\|GJ^{s-1}g\right\|_{p_2}$. We see that by using Lemma A.3.3 from Section A.3 we have

$$\left|\frac{\partial^m\left[(1+|\eta|^2)^{1/2-s/2}\right]}{\partial_{\eta_1}^{\alpha_1}\dots\partial_{\eta_d}^{\alpha_d}}\right| \leqslant C(1+|\eta|^2)^{1/2-s/2-m/2} \qquad \forall\eta\in\mathbb{R}^d. \tag{A.31}$$

Similarly by virtue of Lemma A.3.3 from Section A.3 we have

$$\left|\frac{\partial^m\left[(1+|\eta|^2)^{s/2}\right]}{\partial_{\eta_1}^{\alpha_1}\dots\partial_{\eta_d}^{\alpha_d}}\right| \leqslant C(1+|\eta|^2)^{s/2-m/2} \qquad \forall\eta\in\mathbb{R}^d. \tag{A.32}$$

Also, we have

$$\left|\frac{\partial^m\left[\eta|\eta|^{-2}\right]}{\partial_{\eta_1}^{\alpha_1}\dots\partial_{\eta_d}^{\alpha_d}}\right| \leqslant C|\eta|^{-m-1} \leqslant C\left(1+|\eta|^2\right)^{-\frac{m+1}{2}} \qquad \forall\eta\in\mathbb{R}^d : \frac{1}{1+|\eta|^2} < \frac{9}{10}. \tag{A.33}$$

Applying the same reasoning employed in Lemma A.3.5 from Section A.3 we get

$$\left| \frac{\partial^m \left[\Psi\left(\frac{1}{1+|\eta|^2}\right)\right]}{\partial_{\eta_1}^{\alpha_1} \ldots \partial_{\eta_d}^{\alpha_d}} \right| \leqslant C\left(1+|\eta|^2\right)^{-\frac{m}{2}} \qquad \forall \eta \in \mathbb{R}^d. \tag{A.34}$$

We see that based on (A.31), (A.32), (A.33), (A.34) and Lemma A.3.2 from Section A.3 we get

$$\left| \frac{\partial^m \left[\eta|\eta|^{-2}\left(1+|\eta|^2\right)^{1/2-s/2}\left(\left(1+|\eta|^2\right)^{s/2}-1\right)\Psi\left(\frac{1}{1+|\eta|^2}\right)\right]}{\partial_{\eta_1}^{\alpha_1} \ldots \partial_{\eta_d}^{\alpha_d}} \right| \leqslant C\left(1+|\eta|^2\right)^{-\frac{m}{2}}.$$

From the above, (A.26), (A.27), Lemma A.1.2 and Theorem A.1.1 we have

$$\|Gh\|_{p_2} = \left\| \left(\widehat{(Gh)} \right)^{\vee} \right\|_{p_2} = \left\| \sum_{\eta \in \mathbb{Z}^d} e^{i2\pi\langle \cdot, \eta\rangle} g(\eta)\hat{h}(\eta) \right\|_{p_2} \leqslant C \|h\|_{p_2}.$$

Thus we can conclude based on (A.30) we have

$$\|\sigma_{3,2}(D)(f,g)\|_p \leqslant C \|\partial f\|_{p_1} \left\| J^{s-1} g \right\|_{p_2}. \tag{A.35}$$

Thus using (A.24) and (A.35) we obtain

$$\|\sigma_3(D)(f,g)\|_p \leqslant C\left(\|\partial f\|_{p_1} \left\| J^{s-1} g\right\|_{p_2} + \|g\|_{p_3} \|J^s f\|_{p_4} \right). \tag{A.36}$$

A.2.3. Step 3: Estimate of $\sigma_2(D)(f,g)$

Now we have to estimate

$$\sigma_2(\xi,\eta) = \left(\left(1+|\xi+\eta|^2\right)^{s/2} - \left(1+|\eta|^2\right)^{s/2} \right) \Phi_2\left(\frac{1+|\xi|^2}{1+|\eta|^2} \right).$$

To do this we introduce two new functions

$$\sigma_{2,1}(\xi,\eta) = \left(1+|\xi+\eta|^2\right)^{s/2} \Phi_2\left(\frac{1+|\xi|^2}{1+|\eta|^2} \right), \tag{A.37}$$

$$\sigma_{2,2}(\xi,\eta) = \left(1+|\eta|^2\right)^{s/2} \Phi_2\left(\frac{1+|\xi|^2}{1+|\eta|^2} \right). \tag{A.38}$$

It is clear that $\sigma_2 = \sigma_{2,1} - \sigma_{2,2}$. We see that the following holds:

$$\begin{aligned}\sigma_{2,2}(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta) &= \left(1+|\eta|^2\right)^{s/2}\left(1+|\xi|^2\right)^{-s/2}\Phi_2\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)(J^s f)\widehat{\ }(\xi)\hat{g}(\eta)\\ &= \overline{\sigma}_{2,2}(\xi,\eta)(J^s f)\widehat{\ }(\xi)\hat{g}(\eta).\end{aligned}$$

Recalling that $\operatorname{supp}\Phi_2 \subset \left(\frac{1}{10}, 10\right)$ and using Lemmas A.3.3, A.3.5, A.3.2 from Section A.3 combined with Lemma A.1.2 and Theorem A.1.1 it is easy to see that

$$\left\|\sigma_{2,2}(D)(f,g)\right\|_p = \left\|\overline{\sigma}_{2,2}(D)(J^s f, g)\right\|_p \leqslant C\left\|g\right\|_{p_3}\left\|J^s f\right\|_{p_4}. \tag{A.39}$$

Now we have to provide the estimate for $\sigma_{2,1}$:

$$\begin{aligned}\sigma_{2,1}(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta) &= \left(1+|\xi+\eta|^2\right)^{s/2}\left(1+|\xi|^2\right)^{-s/2}\Phi_2\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)(J^s f)\widehat{\ }(\xi)\hat{g}(\eta)\\ &= \overline{\sigma}_{2,1}(\xi,\eta)(J^s f)\widehat{\ }(\xi)\hat{g}(\eta).\end{aligned} \tag{A.40}$$

As we see in the formulation of Theorem A.1.1, condition (A.1) has to be valid up to some number of differences taken. Let us denote this number by $k(d)$. Now, let us analyse the case where $s/2 \geqslant k(d)$. We try to proceed in the case of $\overline{\sigma}_{2,1}$ in the same way as in the case of $\overline{\sigma}_{2,2}$. Thus we try to validate the assumption (A.2) in Lemma A.1.2. While doing so we may have to estimate negative powers of the term $1+|\xi+\eta|^2$, which is problematic. This is not the issue when $s/2 \geqslant k(d)$ and calculations can be performed similarly to $\overline{\sigma}_{2,2}$ (thanks to Lemma A.3.6 from Section A.3). We will apply the complex interpolation method to obtain the estimate in the case where s is not so large. Thus we extend the definition of $\overline{\sigma}_{2,1}$, $\sigma_{2,1}$ to complex values:

$$\overline{\sigma}^z_{2,1}(\xi,\eta) = \left(1+|\xi+\eta|^2\right)^{z/2}\left(1+|\xi|^2\right)^{-z/2}\Phi_2\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right), \tag{A.41}$$

$$\sigma^z_{2,1}(\xi,\eta) = \left(1+|\xi+\eta|^2\right)^{z/2}\Phi_2\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right), \tag{A.42}$$

such that $0 \leqslant \operatorname{Re} z \leqslant 2k(d)$. If we choose $z = 2k + it$ we can conclude using (A.41), Lemmas A.3.3, A.3.5, A.3.6, A.3.2, A.1.2 and Theorem A.1.1, that for $\psi, \phi \in C^\infty(\mathbb{T}^d)$ we have

$$\left\|\overline{\sigma}^{2k+it}_{2,1}(D)(\phi,\psi)\right\|_p \leqslant C(t)\left\|\psi\right\|_{p_3}\left\|\phi\right\|_{p_4}, \tag{A.43}$$

where $C(t) = \overline{C} \cdot (1 + |t|)^k$ (this factor is the result of $k(d)$ differentiations present in Theorem A.1.1). Now we need to establish a similar estimate in the case of $z = it$. To this end, we observe that (based on transformations that lead to (A.8)) we have

$$J^{it}(\phi\psi) = \sum_{i=1}^{3} \kappa_i^{it}(D)(\phi, \psi),$$

where

$$\kappa_j^{it}(\xi, \eta) = \left(1 + |\xi + \eta|^2\right)^{it/2} \Phi_j \left(\frac{1 + |\xi|^2}{1 + |\eta|^2}\right).$$

We want to obtain an estimate of $\overline{\sigma}_{2,1}^{it}(D)(\phi, \psi)$, however, it is easier to start with obtaining an estimate for $\sigma_{2,1}^{it}(D)(\phi, \psi)$:

$$\sigma_{2,1}^{it}(D)(\phi, \psi) = \kappa_2^{it}(D)(\phi, \psi) = J^{it}(\phi\psi) - \kappa_1^{it}(D)(\phi, \psi) - \kappa_3^{it}(D)(\phi, \psi). \tag{A.44}$$

Now we need to derive estimates for each term on the right-hand side. First we will concentrate on $\kappa_j^{it}(\xi, \eta)$ for $j = 1, 3$. It follows from Lemma A.3.6 from Section A.3 that

$$\begin{aligned} &\left| \frac{\partial^m \left[(1 + |\xi + \eta|^2)^{it/2}\right]}{\partial_{\eta_1}^{\alpha_1} \dots \partial_{\eta_d}^{\alpha_d} \partial_{\eta_1}^{\beta_1} \dots \partial_{\eta_d}^{\beta_d}} \right| \\ &\qquad \leqslant C \cdot (1 + |t|)^m \left(1 + |\eta|^2 + |\xi|^2\right)^{-\frac{m}{2}} \text{ for } \frac{1 + |\xi|^2}{1 + |\eta|^2} > 9 \text{ or } \frac{1 + |\xi|^2}{1 + |\eta|^2} < \frac{1}{9}. \end{aligned}$$

By Lemma A.3.5 from Section A.3 we have

$$\left| \frac{\partial^m \left[\Phi_j \left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)\right]}{\partial_{\eta_1}^{\alpha_1} \dots \partial_{\eta_d}^{\alpha_d} \partial_{\eta_1}^{\beta_1} \dots \partial_{\eta_d}^{\beta_d}} \right| \leqslant C \left(1 + |\eta|^2 + |\xi|^2\right)^{-\frac{m}{2}}.$$

Thus Lemma A.3.2 from Section A.3 implies

$$\left| \frac{\partial^m \left[\kappa_j^{it}(\xi, \eta)\right]}{\partial_{\eta_1}^{\alpha_1} \dots \partial_{\eta_d}^{\alpha_d} \partial_{\eta_1}^{\beta_1} \dots \partial_{\eta_d}^{\beta_d}} \right| \leqslant C \cdot (1 + |t|)^m \left(1 + |\eta|^2 + |\xi|^2\right)^{-\frac{m}{2}} \text{ for } j = 1, 3.$$

Based on Lemma A.1.2 and Theorem A.1.1 we have

$$\left\|\kappa_j^{it}(D)(\phi, \psi)\right\|_p \leqslant C(1 + |t|)^k \left\|\psi\right\|_{p_3} \left\|\phi\right\|_{p_4} \text{ for } j = 1, 3. \tag{A.45}$$

Now we will provide an estimate for $J^{it}(\phi\psi)$. We see that for $h \in C^\infty(\mathbb{T}^d)$ we have

$$J^{it}(h)(x) = \sum_{\eta\in\mathbb{Z}^d} e^{2\pi i\langle x,\eta\rangle} \left(1+|\eta|^2\right)^{it/2} \hat{h}(\eta).$$

We see that in view of Lemma A.3.3 from Section A.3 symbol $\left(1+|\eta|^2\right)^{it/2}$ fulfils assumptions of Lemma A.1.2 and thus the assumption of Theorem A.1.1. Consequently, we have

$$\left\|J^{it}h\right\|_p \leqslant C(1+|t|)^k \left\|h\right\|_p . \tag{A.46}$$

From this we easily get

$$\left\|J^{it}(\phi\psi)\right\|_p \leqslant C(1+|t|)^k \left\|\psi\right\|_{p_3} \left\|\phi\right\|_{p_4} . \tag{A.47}$$

Thus we see that by (A.44), (A.45), (A.47) we have

$$\left\|\sigma^{it}_{2,1}(D)(\phi,\psi)\right\|_p \leqslant C(1+|t|)^k \left\|\psi\right\|_{p_3} \left\|\phi\right\|_{p_4} .$$

Based on (A.41), (A.42) and (A.46) we have

$$\left\|\overline{\sigma}^{it}_{2,1}(D)(\phi,\psi)\right\|_p = \left\|\sigma^{it}_{2,1}(D)(J^{-it}\phi,\psi)\right\|_p \leqslant C(1+|t|)^{2k} \left\|\psi\right\|_{p_3} \left\|\phi\right\|_{p_4} .$$

In order to use Theorem A.1.3 we need to show that the family of operators $\left\{\overline{\sigma}^z_{2,1}(D)\right\}_{z\in\overline{S}}$ is an admissible analytic family. According to the Definition A.1.5 we can verify the conditions for smooth functions (which are dense in L^p, $1<p<\infty$). Let us choose two functions $\psi,\phi \in C^\infty(\mathbb{T}^d)$. We clearly see that

$$S \ni z \longmapsto \sum_{\substack{\xi\in\mathbb{Z}^d:|k|<n\\ \eta\in\mathbb{Z}^d:|k|<n}} e^{2\pi ix(\xi+\eta)} \left(1+|\xi+\eta|^2\right)^{z/2} \left(1+|\xi|^2\right)^{-z/2} \Phi_2\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right) \hat{\phi}(\xi)\hat{\psi}(\eta)$$

is analytic, because functions of type $S \ni z \mapsto \beta^{z/2}$, $\beta \in \mathbb{R}_+$ are analytic. We will show that the expression on the right-hand side converges uniformly. Indeed, we see that using

(A.5) we have

$$\begin{aligned}&\sum_{\xi,\eta\in\mathbb{Z}^d}\left|e^{2\pi ix(\xi+\eta)}\left(1+|\xi+\eta|^2\right)^{z/2}\left(1+|\xi|^2\right)^{-z/2}\Phi_2\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)\hat{\phi}(\xi)\hat{\psi}(\eta)\right|\\&\quad\leqslant C\sum_{\xi\in\mathbb{Z}^d}\left|\hat{\phi}(\xi)\right|\sum_{\eta\in\mathbb{Z}^d}\left|\hat{\psi}(\eta)\right|\leqslant C\sum_{\xi\in\mathbb{Z}^d}\frac{C_\phi}{(1+|\xi|^2)^{\frac{d+1}{2}}}\sum_{\eta\in\mathbb{Z}^d}\frac{C_\psi}{(1+|\xi|^2)^{\frac{d+1}{2}}}<\infty.\end{aligned}\tag{A.48}$$

Thus it is easy to see that

$$\begin{aligned}&\sum_{\substack{\xi\in\mathbb{Z}^d:|k|<n\\ \eta\in\mathbb{Z}^d:|k|<n}}e^{2\pi ix(\xi+\eta)}\left(1+|\xi+\eta|^2\right)^{z/2}\left(1+|\xi|^2\right)^{-z/2}\Phi_2\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)\hat{\phi}(\xi)\hat{\psi}(\eta)\\&\qquad\qquad\overset{n\to\infty}{\rightrightarrows}\ \overline{\sigma}^z_{2,1}(D)(\phi,\psi).\end{aligned}\tag{A.49}$$

Thus, we can conclude that $\overline{\sigma}^z_{2,1}(D)(\phi,\psi)$ is analytic for any $\phi,\psi\in C^\infty(\mathbb{T}^d)$. Using the same approach we can show continuity of $\overline{S}\ni z\mapsto\overline{\sigma}^z_{2,1}(D)(\phi,\psi)$. We will only apply Theorem A.1.3 to one of the variables of $\overline{\sigma}^z_{2,1}(D)(\phi,\psi)$. To show that condition (A.4) holds, we verify that $C(1+|t|)^{2k}\left\|\psi\right\|_{p_3}\in L^1(P_j(\theta,\cdot)dt)$ for $j=0,1$ (the interpolation with respect to the first variable). It is obvious based on Remark A.1.4. Thus using Theorem A.1.3 we can deduce that for $0\leqslant s\leqslant 2k$ the following holds

$$\left\|\overline{\sigma}^s_{2,1}(D)(\phi,\psi)\right\|_p\leqslant C\left\|\psi\right\|_{p_3}\left\|\phi\right\|_{p_4}.$$

Now recalling (A.37), (A.40) and (A.41) we have

$$\left\|\sigma_{2,1}(D)(f,g)\right\|_p=\left\|\overline{\sigma}^s_{2,1}(D)(J^sf,g)\right\|_p\leqslant C\left\|g\right\|_{p_3}\left\|J^sf\right\|_{p_4}.$$

The validity of the above inequality in case $s/2>k$ was already justified in reasoning that lead to (A.43). Thus using the above and (A.39) we obtain

$$\left\|\sigma_2(D)(f,g)\right\|_p\leqslant C\left\|g\right\|_{p_3}\left\|J^sf\right\|_{p_4}.\tag{A.50}$$

A.2.4. Conclusion

By combining (A.8), (A.16), (A.36) and (A.50) we get

$$\|J^s(fg) - fJ^s(g)\|_p \leqslant C\left(\|\partial f\|_{p_1}\left\|J^{s-1}g\right\|_{p_2} + \|g\|_{p_3}\|J^s f\|_{p_4}\right).$$

□

A.3. Auxiliary lemmas

The following lemmas were used in the proof of Lemma 1.3.3.

Lemma A.3.1. *Let ξ, $\eta \in \mathbb{R}^d$ such that $\frac{1+|\xi|^2}{1+|\eta|^2} < \frac{1}{9}$, then*

$$\left|(1+|\eta|^2)^{-1}\langle\xi, \xi + 2\eta\rangle\right| < \frac{7}{9}. \tag{A.51}$$

Proof of Lemma A.3.1. We have

$$\left|(1+|\eta|^2)^{-1}\langle\xi, \xi + 2\eta\rangle\right| \leqslant \frac{|\xi|^2 + 2|\langle\xi,\eta\rangle|}{1+|\eta|^2} \leqslant \frac{|\xi|^2}{1+|\eta|^2} + 2\frac{|\xi|}{\sqrt{1+|\eta|^2}}\frac{|\eta|}{\sqrt{1+|\eta|^2}}.$$

Using the fact that $\frac{1+|\xi|^2}{1+|\eta|^2} < \frac{1}{9}$, we have

$$\left|(1+|\eta|^2)^{-1}\langle\xi, \xi + 2\eta\rangle\right| < \frac{1}{9} + 2\cdot\sqrt{\frac{1}{9}}\cdot 1 = \frac{7}{9}.$$

□

Lemma A.3.2. *Let $N \in \mathbb{N}$, $d \in \mathbb{N}_+$. Suppose that $\sigma_1, \sigma_2 : \mathbb{R}^d \to \mathbb{C}$ are two symbols satisfying*

$$\left|\partial_\xi^\alpha \sigma_i(\xi)\right| \leqslant \frac{C_\alpha^i F_i(\xi)}{(1+|\xi|^2)^{\frac{|\alpha|}{2}}} \qquad \forall \xi \in U \subset \mathbb{R}^d \tag{A.52}$$

for all $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ such that $|\alpha| \leqslant N$ and $F_i : \mathbb{R}^d \to \mathbb{R}_{\geqslant 0}$. Let us define $\sigma = \sigma_1\sigma_2$. Then for all $\alpha \in \mathbb{N}^d$ such that $|\alpha| \leqslant N$ there exists constant C_α such that

$$\left|\partial_\xi^\alpha \sigma(\xi)\right| \leqslant \frac{C_\alpha F_1(\xi)F_2(\xi)}{(1+|\xi|^2)^{\frac{|\alpha|}{2}}} \qquad \forall \xi \in U \subset \mathbb{R}^d.$$

Proof of Lemma A.3.2. Let us set $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ such that $|\alpha| \leqslant N$. Now we calculate

$$\frac{\partial^{|\alpha|}[\sigma_1\sigma_2]}{\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_d}^{\alpha_d}} = \sum_{k_1=0}^{\alpha_1}\dots\sum_{k_d=0}^{\alpha_d}\binom{\alpha_1}{k_1}\dots\binom{\alpha_d}{k_d}\frac{\partial^{\sum_{i=1}^d k_i}[\sigma_1]}{\partial_{\xi_1}^{k_1}\dots\partial_{\xi_d}^{k_d}}\frac{\partial^{|\alpha|-\sum_{i=1}^d k_i}[\sigma_2]}{\partial_{\xi_1}^{\alpha_1-k_1}\dots\partial_{\xi_d}^{\alpha_d-k_d}}.$$

By the assumption (A.52) we get

$$\begin{aligned}\left|\frac{\partial^{|\alpha|}[\sigma_1\sigma_2]}{\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_d}^{\alpha_d}}\right| &\leqslant \sum_{k_1=0}^{\alpha_1}\dots\sum_{k_d=0}^{\alpha_d}\binom{\alpha_1}{k_1}\dots\binom{\alpha_d}{k_d}\frac{C_{\alpha_1,\dots,\alpha_d}F_1(\xi)}{(1+|\xi|^2)^{\frac{\sum_{i=1}^d k_i}{2}}}\frac{C_{\alpha_1-k_1,\dots,\alpha_d-k_d}F_2(\xi)}{(1+|\xi|^2)^{|\alpha|-\frac{\sum_{i=1}^d k_i}{2}}}\\ &= \frac{F_1(\xi)F_2(\xi)}{(1+|\xi|^2)^{\frac{|\alpha|}{2}}}\sum_{k_1=0}^{\alpha_1}\dots\sum_{k_d=0}^{\alpha_d}\binom{\alpha_1}{k_1}\dots\binom{\alpha_d}{k_d}C_{\alpha_1-k_1,\dots,\alpha_d-k_d}C_{\alpha_1,\dots,\alpha_d}.\end{aligned}$$

□

Lemma A.3.3. *Let $s \in \mathbb{C}$, $m \in \mathbb{N}$ and $\{\alpha_i\}_{i=1}^d \in \mathbb{N}^d$ such that $\sum_{i=1}^d \alpha_i = m$. Then, there exist $N \in \mathbb{N}$, $\{\omega_{i,j}\}_{i,j=1}^{N,d} \in \mathbb{N}^{dN}$, $\{k_i\}_{i=1}^N \in \mathbb{N}^N$, $\{C_i\}_{i=1}^N \in \mathbb{C}^N$ such that*

$$\frac{\partial^m\left[(1+|\eta|^2)^s\right]}{\partial_{\eta_1}^{\alpha_1}\dots\partial_{\eta_d}^{\alpha_d}} = \sum_{i=1}^N C_i(1+|\eta|^2)^{s-k_i}\eta_1^{\omega_{i,1}}\dots\eta_d^{\omega_{i,d}}, \tag{A.53}$$

where $\forall i \in \{1,\dots,N\}$ $0 \leqslant k_i \leqslant m$, $2k_i - \sum_{j=1}^d \omega_{i,j} = m$ and $|C_i(s,\alpha_1,\dots,\alpha_d)| \leqslant C(m)\cdot(1+|s|)^m$. Also, we have

$$\left|\frac{\partial^m\left[(1+|\eta|^2)^s\right]}{\partial_{\eta_1}^{\alpha_1}\dots\partial_{\eta_d}^{\alpha_d}}\right| \leqslant C(m)(1+|s|)^m(1+|\eta|^2)^{\operatorname{Re}s-\frac{m}{2}}.$$

Proof of Lemma A.3.3. We will prove the representation formula (A.53) using the induction method. Let us observe that

$$\frac{\partial\left[(1+|\eta|^2)^s\right]}{\partial_{\eta_i}} = 2s(1+|\eta|^2)^{s-1}\eta_i$$

and thus formula (A.53) holds for one differentiation. Now we assume that it holds for a certain number of differentiations and will try to deduce its validity after additional differentiation. Indeed we have

$$\begin{aligned}\frac{\partial^{m+1}\left[(1+|\eta|^2)^s\right]}{\partial_{\eta_1}^{\alpha_1}\dots\partial_{\eta_j}^{\alpha_j+1}\dots\partial_{\eta_d}^{\alpha_d}} &= \frac{\partial}{\partial_{\eta_j}}\sum_{i=1}^{N_m} C_i(1+|\eta|^2)^{s-k_i}\eta_1^{\omega_{i,1}}\dots\eta_d^{\omega_{i,d}}\\ &= \sum_{i=1}^{N_m} 2C_i(s-k_i)(1+|\eta|^2)^{s-(k_i+1)}\eta_1^{\omega_{i,1}}\dots\eta_j^{\omega_{i,j}+1}\dots\eta_d^{\omega_{i,d}}\\ &+\sum_{i=1}^{N_m} C_i(1+|\eta|^2)^{s-k_i}\eta_1^{\omega_{i,1}}\dots\eta_j^{\omega_{i,j}-1}\dots\eta_d^{\omega_{i,d}}.\end{aligned}$$

We observe that $2(k_i+1)-\sum_{j=1}^d\omega_{i,j}-1=m+1$ and $|C_i(s-k_i)|\lesssim\widetilde{C}(1+|s|)^{m+1}$. Thus we proved that (A.53) holds. Now it is easy to verify that

$$\begin{aligned}\left|\frac{\partial^m\left[(1+|\eta|^2)^s\right]}{\partial_{\eta_1}^{\alpha_1}\dots\partial_{\eta_d}^{\alpha_d}}\right| &\leqslant C(1+|s|)^m(1+|\eta|^2)^{\operatorname{Re}s}\sum_{i=1}^{N_m}\frac{|\eta|^{\sum_{j=1}^d\omega i,j}}{(1+|\eta|^2)^{k_i}}\\ &\leqslant C(1+|s|)^m(1+|\eta|^2)^{\operatorname{Re}s}\sum_{i=1}^{N_m}(1+|\eta|^2)^{-k_i+\frac{1}{2}\sum_{j=1}^d\omega i,j}\\ &\leqslant C(1+|s|)^m(1+|\eta|^2)^{\operatorname{Re}s-\frac{m}{2}}.\end{aligned}$$

□

Lemma A.3.4. *Let $r\in\mathbb{N}_+$, $m\in\mathbb{N}$ and $\{\alpha_i\}_{i=1}^d,\{\beta_i\}_{i=1}^d\in\mathbb{N}^d$ such that $\sum_{i=1}^d(\alpha_i+\beta_i)=m$. Then, there exist $N\in\mathbb{N}$, $\{\omega_{i,j}\}_{i,j=1}^{N,2d}\in\mathbb{N}^{N\times 2d}$, $\{k_i\}_{i=1}^N\in\mathbb{N}^N$, $\{C_i\}_{i=1}^N\in\mathbb{R}^N$ such that*

$$\frac{\partial^m\left[\langle\xi,\xi+2\eta\rangle^{r-1}(\xi_k+2\eta_k)\right]}{\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_d}^{\alpha_d}\partial_{\eta_1}^{\beta_1}\dots\partial_{\eta_d}^{\beta_d}}=\sum_{i=1}^N C_i\langle\xi,\xi+2\eta\rangle^{r-1-k_i}\xi_1^{\omega_{i,1}}\dots\xi_d^{\omega_{i,d}}\eta_1^{\omega_{i,d+1}}\dots\eta_d^{\omega_{i,2d}}. \quad (A.54)$$

Moreover, for $i=1,\dots,N$ we have $0\leqslant k_i\leqslant m$, $r-1-k_i\geqslant 0$, $2k_i-\sum_{j=1}^{2d}\omega_{i,j}=m-1$ and $|C_i|\leqslant C(1+r)^m$. Additionally, there exists C independent of r such that for $(\xi,\eta):\frac{1+|\xi|^2}{1+|\eta|^2}<\frac{1}{9}$ we have

$$\left|\frac{\partial^m\left[\langle\xi,\xi+2\eta\rangle^{r-1}(\xi_k+2\eta_k)\right]}{\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_d}^{\alpha_d}\partial_{\eta_1}^{\beta_1}\dots\partial_{\eta_d}^{\beta_d}}\right|\leqslant C(1+r)^m\left(\frac{7}{9}\right)^r\frac{(1+|\eta|^2)^{r-\frac{1}{2}}}{(1+|\xi|^2+|\eta|^2)^{\frac{m}{2}}}.$$

Proof of Lemma A.3.4. We will prove this statement via induction argument. We see that for one derivative we have

$$\frac{\partial\left[\langle\xi,\xi+2\eta\rangle^{r-1}(\xi_k+2\eta_k)\right]}{\partial_{\xi_j}} = (r-1)\langle\xi,\xi+2\eta\rangle^{r-2}(2\xi_j+2\eta_j)(\xi_k+2\eta_k) + \langle\xi,\xi+2\eta\rangle^{r-1}\delta_{kj},$$

$$\frac{\partial\left[\langle\xi,\xi+2\eta\rangle^{r-1}(\xi_k+2\eta_k)\right]}{\partial_{\eta_j}} = (r-1)\langle\xi,\xi+2\eta\rangle^{r-2}2\xi_j(\xi_k+2\eta_k) + 2\langle\xi,\xi+2\eta\rangle^{r-1}\delta_{kj}.$$

We see that the above results of differentiations match the form of (A.54). Now, we make the induction step

$$\begin{aligned}
&\frac{\partial^{m+1}\left[\langle\xi,\xi+2\eta\rangle^{r-1}(\xi_k+2\eta_k)\right]}{\partial^{\alpha_1}_{\xi_1}\dots\partial^{\alpha_j+1}_{\xi_j}\dots\partial^{\alpha_d}_{\xi_d}\partial^{\beta_1}_{\eta_1}\dots\partial^{\beta_d}_{\eta_d}} = \frac{\partial}{\partial_{\xi_j}}\left(\sum_{i=1}^{N_m} C_i\langle\xi,\xi+2\eta\rangle^{r-1-k_i}\prod_{k=1}^{d}\xi_k^{\omega_{i,k}}\eta_k^{\omega_{i,d+k}}\right)\\
&= \sum_{i=1}^{N_m} 2C_i(r-1-k_i)\langle\xi,\xi+2\eta\rangle^{r-1-(k_i+1)}(\xi_j+\eta_j)\xi_1^{\omega_{i,1}}\dots\xi_d^{\omega_{i,d}}\eta_1^{\omega_{i,d+1}}\dots\eta_d^{\omega_{i,2d}}\\
&\quad+\sum_{i=1}^{N_m} C_i\langle\xi,\xi+2\eta\rangle^{r-1-k_i}\xi_1^{\omega_{i,1}}\dots\xi_j^{\omega_{i,j}-1}\dots\xi_d^{\omega_{i,d}}\eta_1^{\omega_{i,d+1}}\dots\eta_d^{\omega_{i,2d}}.
\end{aligned}$$

We see that $2(k_i+1)-(\sum_{j=1}^{2d}\omega_{i,j}+1) = (m+1)-1$ and $2k_i-(\sum_{j=1}^{2d}\omega_{i,j}-1) = (m+1)-1$, thus postulated equality (A.54) holds. In the same way we get equality for $\frac{\partial}{\partial_{\eta_j}}$.
Now, we will prove the inequality stated in the lemma. We see that

$$\left|\frac{\partial^{m}\left[\langle\xi,\xi+2\eta\rangle^{r-1}(\xi_k+2\eta_k)\right]}{\partial^{\alpha_1}_{\xi_1}\dots\partial^{\alpha_d}_{\xi_d}\partial^{\beta_1}_{\eta_1}\dots\partial^{\beta_d}_{\eta_d}}\right| \leqslant C(1+r)^m\sum_{i=1}^{N_m}|\langle\xi,\xi+2\eta\rangle|^{r-1-k_i}|\xi|^{\sum_{j=1}^{d}\omega_{i,j}}|\eta|^{\sum_{j=n+1}^{2d}\omega_{i,j}}.$$

We modify the right-hand side in the following way

$$\begin{aligned}
&\left|\frac{\partial^{m}\left[\langle\xi,\xi+2\eta\rangle^{r-1}(\xi_k+2\eta_k)\right]}{\partial^{\alpha_1}_{\xi_1}\dots\partial^{\alpha_d}_{\xi_d}\partial^{\beta_1}_{\eta_1}\dots\partial^{\beta_d}_{\eta_d}}\right|\\
&\leqslant C(m)(1+r)^m\sum_{i=1}^{N_m}\left(\frac{9}{7}\frac{|\langle\xi,\xi+2\eta\rangle|}{1+|\eta|^2}\right)^{r-1-k_i}\left(\frac{7}{9}(1+|\eta|^2)\right)^{r-1-k_i}|\xi|^{\sum_{j=1}^{d}\omega_{i,j}}|\eta|^{\sum_{j=d+1}^{2d}\omega_{i,j}}.
\end{aligned}$$

Based on the fact that $r-1-k_i \geqslant 0$ and on Lemma A.3.1 we have

$$\left|\frac{\partial^m\left[\langle\xi,\xi+2\eta\rangle^{r-1}(\xi_k+2\eta_k)\right]}{\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_d}^{\alpha_d}\partial_{\eta_1}^{\beta_1}\dots\partial_{\eta_d}^{\beta_d}}\right| \leqslant C(m)(1+r)^m\left(\frac{7}{9}\right)^r\left(1+|\eta|^2\right)^{r-\frac{1}{2}}\sum_{i=1}^{N_m}\frac{|\xi|^{\sum_{j=1}^{d}\omega_{i,j}}|\eta|^{\sum_{j=d+1}^{2d}\omega_{i,j}}}{\left(1+|\eta|^2\right)^{k_i+\frac{1}{2}}}.$$

Using $\frac{1+|\xi|^2}{1+|\eta|^2}<\frac{1}{9}$ we have

$$\left|\frac{\partial^m\left[\langle\xi,\xi+2\eta\rangle^{r-1}(\xi_k+2\eta_k)\right]}{\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_d}^{\alpha_d}\partial_{\eta_1}^{\beta_1}\dots\partial_{\eta_d}^{\beta_d}}\right| \leqslant C(m)(1+r)^m\left(\frac{7}{9}\right)^r\left(1+|\eta|^2\right)^{r-\frac{1}{2}}\sum_{i=1}^{N_m}\frac{\left(1+|\xi|^2+|\eta|^2\right)^{\frac{1}{2}\sum_{j=1}^{2d}\omega_{i,j}}}{\left(1+|\xi|^2+|\eta|^2\right)^{k_i+\frac{1}{2}}}.$$

Now using the fact that $2k_i-\sum_{j=1}^{2d}\omega_{i,j}=m-1$ we get

$$\left|\frac{\partial^m\left[\langle\xi,\xi+2\eta\rangle^{r-1}(\xi_k+2\eta_k)\right]}{\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_d}^{\alpha_d}\partial_{\eta_1}^{\beta_1}\dots\partial_{\eta_d}^{\beta_d}}\right| \leqslant C(m)(1+r)^m\left(\frac{7}{9}\right)^r\frac{\left(1+|\eta|^2\right)^{r-\frac{1}{2}}}{\left(1+|\xi|^2+|\eta|^2\right)^{\frac{m}{2}}}.$$

□

Lemma A.3.5. *Let $m\in\mathbb{N}_+$ and let $\{\alpha_i\}_{i=1}^d,\{\beta_i\}_{i=1}^d\in\mathbb{N}^d$ be such that $\sum_{i=1}^d(\alpha_i+\beta_i)=m$. Let $\Phi\in C^\infty(\mathbb{R})$ be such that $\operatorname{supp}\frac{\partial\Phi}{\partial x}\subset[a,b]$ for some $a,b>0$. Then we have*

$$\left|\frac{\partial^m\left[\Phi\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)\right]}{\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_d}^{\alpha_d}\partial_{\eta_1}^{\beta_1}\dots\partial_{\eta_d}^{\beta_d}}\right| \leqslant C\left(1+|\xi|^2+|\eta|^2\right)^{-\frac{m}{2}}. \tag{A.55}$$

Proof of Lemma A.3.5. First, we will show that there exist $N\in\mathbb{N}$, $\{\omega_{i,j}\}_{i,j=1}^{N,2d}\in\mathbb{N}^{N\times 2d}$, $\{k_i\}_{i=1}^N,\{\kappa_{i,\xi}\}_{i=1}^N,\{\kappa_{i,\eta}\}_{i=1}^N\in\mathbb{N}^N$, $\{C_i\}_{i=1}^N\in\mathbb{R}^N$ such that derivatives can be expressed in the following way:

$$\frac{\partial^m\left[\Phi\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)\right]}{\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_d}^{\alpha_d}\partial_{\eta_1}^{\beta_1}\dots\partial_{\eta_d}^{\beta_d}} = \sum_{i=1}^{N_m}C_i\Phi^{(k_i)}\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)\frac{(1+|\xi|^2)^{\kappa_{i,\eta}}}{(1+|\eta|^2)^{\kappa_{i,\xi}+\kappa_{i,\eta}}}\xi_1^{\omega_{i,1}}\dots\xi_d^{\omega_{i,d}}\eta_1^{\omega_{i,d+1}}\dots\eta_d^{\omega_{i,2d}}, \tag{A.56}$$

where $1\leqslant k_i\leqslant m$, $2\kappa_{i,\xi}-\sum_{j=1}^{2d}\omega_{i,j}=m$. We see that for $m=1$ such a representation is valid. Now we assume that the formula holds for some number of differentiations and we will check if the formula is still valid after additional differentiation. We start with the differentiation with respect to ξ_j:

$$\frac{\partial^{m+1}\left[\Phi\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)\right]}{\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_j}^{\alpha_j+1}\dots\partial_{\xi_d}^{\alpha_d}\partial_{\eta_1}^{\beta_1}\dots\partial_{\eta_d}^{\beta_d}}$$
$$= \frac{\partial}{\partial_{\xi_j}}\left[\sum_{i=1}^{N_m} C_i \Phi^{(k_i)}\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)\frac{(1+|\xi|^2)^{\kappa_{i,\eta}}}{(1+|\eta|^2)^{\kappa_{i,\xi}+\kappa_{i,\eta}}}\xi_1^{\omega_{i,1}}\dots\xi_d^{\omega_{i,d}}\eta_1^{\omega_{i,d+1}}\dots\eta_d^{\omega_{i,2d}}\right].$$

After carrying out the differentiation we get

$$\frac{\partial^{m+1}\left[\Phi\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)\right]}{\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_j}^{\alpha_j+1}\dots\partial_{\xi_d}^{\alpha_d}\partial_{\eta_1}^{\beta_1}\dots\partial_{\eta_d}^{\beta_d}}$$
$$= \sum_{i=1}^{N_m} 2C_i \Phi^{(k_i+1)}\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)\frac{(1+|\xi|^2)^{\kappa_{i,\eta}}}{(1+|\eta|^2)^{\kappa_{i,\xi}+\kappa_{i,\eta}+1}}\xi_j\prod_{k=1}^{d}\xi_k^{\omega_{i,k}}\eta_k^{\omega_{i,d+k}}$$
$$+ \sum_{i=1}^{N_m} C_i 2\kappa_{i,\eta} \Phi^{(k_i)}\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)\frac{(1+|\xi|^2)^{\kappa_{i,\eta}-1}}{(1+|\eta|^2)^{\kappa_{i,\xi}+1+\kappa_{i,\eta}-1}}\xi_j\prod_{k=1}^{d}\xi_k^{\omega_{i,k}}\eta_k^{\omega_{i,d+k}}$$
$$+ \sum_{i=1}^{N_m} C_i \omega_{i,j} \Phi^{(k_i)}\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)\frac{(1+|\xi|^2)^{\kappa_{i,\eta}}}{(1+|\eta|^2)^{\kappa_{i,\xi}+\kappa_{i,\eta}}}\xi_j^{-1}\prod_{k=1}^{d}\xi_k^{\omega_{i,k}}\eta_k^{\omega_{i,d+k}}.$$

The obtained formula matches the structure from equation (A.56). Now let us check the validity after the additional differentiation with respect to η_j:

$$\frac{\partial^{m+1}\left[\Phi\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)\right]}{\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_d}^{\alpha_d}\partial_{\eta_1}^{\beta_1}\dots\partial_{\eta_j}^{\beta_j+1}\dots\partial_{\eta_d}^{\beta_d}}$$
$$= \frac{\partial}{\partial_{\eta_j}}\left[\sum_{i=1}^{N_m} C_i \Phi^{(k_i)}\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)\frac{(1+|\xi|^2)^{\kappa_{i,\eta}}}{(1+|\eta|^2)^{\kappa_{i,\xi}+\kappa_{i,\eta}}}\xi_1^{\omega_{i,1}}\dots\xi_d^{\omega_{i,d}}\eta_1^{\omega_{i,d+1}}\dots\eta_d^{\omega_{i,2d}}\right].$$

After carrying out the differentiation we get

$$\frac{\partial^{m+1}\left[\Phi\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)\right]}{\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_d}^{\alpha_d}\partial_{\eta_1}^{\beta_1}\dots\partial_{\eta_j}^{\beta_j+1}\dots\partial_{\eta_d}^{\beta_d}}$$
$$= \sum_{i=1}^{N_m}(-2) C_i \Phi^{(k_i+1)}\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)\frac{(1+|\xi|^2)^{\kappa_{i,\eta}+1}}{(1+|\eta|^2)^{\kappa_{i,\xi}+1+\kappa_{i,\eta}+1}}\eta_j\prod_{k=1}^{d}\xi_k^{\omega_{i,k}}\eta_k^{\omega_{i,d+k}}$$
$$+ \sum_{i=1}^{N_m} C_i(-2)(\kappa_{i,\eta}+\kappa_{i,\xi}) \Phi^{(k_i)}\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)\frac{(1+|\xi|^2)^{\kappa_{i,\eta}}}{(1+|\eta|^2)^{\kappa_{i,\xi}+1+\kappa_{i,\eta}}}\eta_j\prod_{k=1}^{d}\xi_k^{\omega_{i,k}}\eta_k^{\omega_{i,d+k}}$$

$$+\sum_{i=1}^{N_m} C_i \omega_{i,d+j} \Phi^{(k_i)} \left(\frac{1+|\xi|^2}{1+|\eta|^2}\right) \frac{(1+|\xi|^2)^{\kappa_{i,\eta}}}{(1+|\eta|^2)^{\kappa_{i,\xi}+\kappa_{i,\eta}}} \eta_j^{-1} \prod_{k=1}^{d} \xi_k^{\omega_{i,k}} \eta_k^{\omega_{i,d+k}}.$$

Again we see that the structure of (A.56) is preserved after differentiation. Now, we can finally prove the estimate (A.55). First, let us observe that

$$\left| \frac{\partial^m \left[\Phi\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)\right]}{\partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_d}^{\alpha_d} \partial_{\eta_1}^{\beta_1} \dots \partial_{\eta_d}^{\beta_d}} \right| = 0 \qquad \text{for } (\xi,\eta) : \frac{1+|\xi|^2}{1+|\eta|^2} \leqslant a \text{ or } \frac{1+|\xi|^2}{1+|\eta|^2} \geqslant b$$

and thus we will focus on the case when $a < \frac{1+|\xi|^2}{1+|\eta|^2} < b$:

$$\begin{aligned} \left| \frac{\partial^m \left[\Phi\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)\right]}{\partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_d}^{\alpha_d} \partial_{\eta_1}^{\beta_1} \dots \partial_{\eta_d}^{\beta_d}} \right| &\leqslant C \sum_{i=1}^{N_m} \frac{(1+|\xi|^2)^{\kappa_{i,\eta}}}{(1+|\eta|^2)^{\kappa_{i,\xi}+\kappa_{i,\eta}}} |\xi_1|^{\omega_{i,1}} \dots \xi_d^{\omega_{i,d}} \eta_1^{\omega_{i,d+1}} \dots \eta_d^{\omega_{i,2d}} \\ &\leqslant C \sum_{i=1}^{N_m} \frac{(1+|\xi|^2+|\eta|^2)^{\frac{1}{2}\sum_{j=1}^{2d} \omega_{i,j}}}{(1+|\eta|^2)^{\kappa_{i,\xi}}}. \end{aligned}$$

Now we use the inequality

$$1+|\eta|^2 \geqslant \frac{1}{2}\left(1+|\eta|^2\right) + \frac{1}{2b}\left(1+|\xi|^2\right) \geqslant \min\left\{\frac{1}{2}, \frac{1}{2b}\right\}\left(1+|\eta|^2+|\xi|^2\right)$$

to obtain

$$\begin{aligned} \left| \frac{\partial^m \left[\Phi\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)\right]}{\partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_d}^{\alpha_d} \partial_{\eta_1}^{\beta_1} \dots \partial_{\eta_d}^{\beta_d}} \right| &\leqslant C \sum_{i=1}^{N_m} \left(1+|\xi|^2+|\eta|^2\right)^{\frac{1}{2}\left(-2\kappa_{1,\xi}+\sum_{j=1}^{2d} \omega_{i,j}\right)} \\ &\leqslant C \left(1+|\xi|^2+|\eta|^2\right)^{-\frac{m}{2}}. \end{aligned}$$

□

Lemma A.3.6. *Let $s \in \mathbb{C}$ be such that $\operatorname{Re} s \geqslant 0$, $m \in \mathbb{N}_+$ and let $\{\alpha_i\}_{i=1}^d, \{\beta_i\}_{i=1}^d \in \mathbb{N}^d$ be such that $\sum_{i=1}^d (\alpha_i+\beta_i) = m$. Then there exist $N \in \mathbb{N}$, $\{\omega_{i,j}\}_{i,j=1}^{N,2d} \in \mathbb{N}^{dN}$, $\{k_i\}_{i=1}^N \in \mathbb{N}^N$, $\{C_i\}_{i=1}^N \in \mathbb{C}^N$ such that*

$$\frac{\partial^m \left[(1+|\xi+\eta|^2)^s\right]}{\partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_d}^{\alpha_d} \partial_{\eta_1}^{\beta_1} \dots \partial_{\eta_d}^{\beta_d}} = \sum_{i=1}^{N} C_i(s)(1+|\xi+\eta|^2)^{s-k_i} \eta_1^{\omega_{i,1}} \dots \eta_d^{\omega_{i,d}} \xi_1^{\omega_{i,d+1}} \dots \xi_d^{\omega_{i,2d}}, \qquad \text{(A.57)}$$

where $\forall i \in \{1,\ldots,N\}$ $0 \leqslant k_i \leqslant m$, $2k_i - \sum_{j=1}^{2d}\omega_{i,j} = m$ *and* $|C_i(s)| \leqslant \overline{C}(1+|s|)^{k_i}$. *Moreover, we have*

$$\left|\frac{\partial^m\left[(1+|\xi+\eta|^2)^s\right]}{\partial_{\xi_1}^{\alpha_1}\ldots\partial_{\xi_d}^{\alpha_d}\partial_{\eta_1}^{\beta_1}\ldots\partial_{\eta_d}^{\beta_d}}\right| \leqslant C(1+|s|)^m(1+|\xi|^2+|\eta|^2)^{\operatorname{Re}s-\frac{m}{2}} \qquad \forall(\xi,\eta)\in U, \tag{A.58}$$

where $U = \left\{(\xi,\eta) : \frac{1+|\xi|^2}{1+|\eta|^2} > 9 \text{ or } \frac{1+|\xi|^2}{1+|\eta|^2} < \frac{1}{9}\right\}$. *If* $\operatorname{Re}s - m \geqslant 0$, *then inequality (A.58) holds for* $(\xi,\eta)\in\mathbb{R}^{2d}$.

Proof of Lemma A.3.6. The representation formula (A.57) can be obtained in the same way as in Lemma A.3.3, thus we will concentrate only on the inequality. We get

$$\left|\frac{\partial^m\left[(1+|\xi+\eta|^2)^s\right]}{\partial_{\xi_1}^{\alpha_1}\ldots\partial_{\xi_d}^{\alpha_d}\partial_{\eta_1}^{\beta_1}\ldots\partial_{\eta_d}^{\beta_d}}\right| \leqslant C(1+|s|)^m\sum_{i=1}^{N_m}\frac{(1+|\eta|^2+|\xi|^2)^{\operatorname{Re}s+\frac{1}{2}\sum_{j=1}^{2d}\omega_{i,j}}}{(1+|\xi+\eta|^2)^{k_i}}. \tag{A.59}$$

Let us observe that from $1+|\xi|^2 > 9\,(1+|\eta|^2)$ we can derive $-\frac{1}{3}\sqrt{|\xi|^2-8} < -|\eta|$. Thus we have

$$\begin{aligned}1+|\xi+\eta|^2 &= 1+|\xi|^2+|\eta|^2+2\langle\xi,\eta\rangle \geqslant 1+|\xi|^2+|\eta|^2-2|\xi||\eta|\\ &\geqslant 1+|\xi|^2+|\eta|^2-\frac{2}{3}|\xi|\sqrt{|\xi|^2-8} \geqslant \frac{1}{3}\left(1+|\xi|^2+|\eta|^2\right).\end{aligned}$$

Hence

$$\left|\frac{\partial^m\left[(1+|\xi+\eta|^2)^s\right]}{\partial_{\eta_1}^{\alpha_1}\ldots\partial_{\eta_d}^{\alpha_d}\partial_{\eta_1}^{\beta_1}\ldots\partial_{\eta_d}^{\beta_d}}\right| \leqslant C(1+|s|)^m\sum_{i=1}^{N_m}\left(1+|\eta|^2+|\xi|^2\right)^{\operatorname{Re}s-\frac{1}{2}(2k_i-\sum_{j=1}^{2d}\omega_{i,j})}.$$

Using the fact that $2k_i - \sum_{j=1}^{2d}\omega_{i,j} = m$, we get the desired inequality. The other case $1+|\xi|^2 < \frac{1}{9}\,(1+|\eta|^2)$ is analogous. Now, let us assume that $\operatorname{Re}s - m \geqslant 0$. Thus from (A.57) we have

$$\left|\frac{\partial^m\left[(1+|\xi+\eta|^2)^s\right]}{\partial_{\xi_1}^{\alpha_1}\ldots\partial_{\xi_d}^{\alpha_d}\partial_{\eta_1}^{\beta_1}\ldots\partial_{\eta_d}^{\beta_d}}\right| \leqslant C(1+|s|)^m\sum_{i=1}^{N}(1+|\xi+\eta|^2)^{\operatorname{Re}s-k_i}\left(1+|\eta|^2+|\xi|^2\right)^{\frac{1}{2}\sum_{j=1}^{2d}\omega_{i,j}}.$$

As $k_i \leqslant m$ we see that $\operatorname{Re}s - k_i \geqslant 0$ and thus we have

$$\left|\frac{\partial^m\left[(1+|\xi+\eta|^2)^s\right]}{\partial_{\xi_1}^{\alpha_1}\ldots\partial_{\xi_d}^{\alpha_d}\partial_{\eta_1}^{\beta_1}\ldots\partial_{\eta_d}^{\beta_d}}\right| \leqslant C(1+|s|)^m\sum_{i=1}^{N}\left(1+|\eta|^2+|\xi|^2\right)^{\operatorname{Re}s-\frac{1}{2}(2k_i-\sum_{j=1}^{2d}\omega_{i,j})}.$$

Again, using the fact that $2k_i - \sum_{j=1}^{2d} \omega_{i,j} = m$, we obtain the desired inequality. $\square$

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